Appendix

A. Proof of Result 1.

For simplicity, we drop the subscript in the following derivation. (Removing the subscript also makes it clear that the result can be applied to both cluster and individual randomized trials.)

Under the monotonicity assumption, we have

$$\tau^R = \sum_{s=a,c} \tau_s P(S = s|R = 1). \qquad (A.1)$$

Because

$$P(S = s|R = 1) = \sum_{z=0,1} P(S = s|R = 1, Z = z)P(Z = z|R = 1), \qquad (A.2)$$

we need to identify the two components $P(S = s|R = 1, Z = z)$ and $P(Z = z|R = 1)$.

Step 1: identify $P(S = s|R = 1, Z = z)$.

First, as we elaborated in the main text, it is straightforward to verify that under monotonicity, the recruited subjects in the control arm are all always-recruited, i.e., $P(S = a|Z = 0, R = 1) = 1$, and $P(S = c|Z = 0, R = 1) = 0$.

The intervention arm consists of always-recruited and compliant-recruited. By definition of conditional probability, for each stratum s, we have

$$P(S = s|Z = 1, R = 1) = \frac{P(R = 1|S = s, Z = 1) \times P(S = s|Z = 1)}{P(R = 1|Z = 1)} \qquad (A.3)$$

For always-recruited and compliant-recruited, we have $P(R = 1|S = s, Z = 1) = 1$. Also, due to randomization in the overall population, we have $P(S = s|Z = 1) = P(S = s)$. Plugging these two equations into Equation (A.3) and take the ratio between always-recruited and compliers, we obtain

$$\frac{P(S=a|Z=1,R=1)}{P(S=c|Z=1,R=1)} = \frac{P(S=a|Z=1)}{P(S=c|Z=1)} = \frac{P(S=a)}{P(S=c)} \qquad (A.4)$$

(A.4) implies that the ratio between the proportion of always-recruited and compliant-recruited in the overall population is the same as the ratio of the proportion of always-recruited and compliant-recruited in the recruited intervention arm. This helps to identify the marginal probabilities of always-recruited and compliant-recruited in the recruited intervention arm (which adds up to 1) given the marginal probabilities of each strata in the overall population. Specifically, let $p_s = P(S=s)$, then we have $P(S=c|Z=1,R=1) = p_c/(p_c+p_a)$, and $P(S=a|Z=1,R=1) = p_a/(p_c+p_a)$.

$$P(S=c|Z=1,R=1) = \frac{p_c}{p_c+p_a}, \qquad P(S=a|Z=1,R=1) = \frac{p_a}{p_c+p_a} \qquad (A.5)$$

Step 2: identify $P(Z=z|R=1)$.

Note that

$$P(Z=z|R=1) = \frac{P(R=1|Z=z)P(Z=z)}{P(R=1)} \qquad (A.6)$$

Because $P(R=1|Z=z) = \sum_{s=a,c} P(R=1|S=s,Z=z)P(S=s|Z=z)$, we have

$$P(R=1|Z=0) = P(R=1|S=a,Z=0)P(S=a|Z=0) = p_a,$$

$$P(R=1|Z=1) = p_a + p_c.$$

Denote the randomization probability as $P(Z = 1) = r$. So, the total recruitment rate is $P(R = 1) = \sum_{z=0,1} P(R = 1|Z = z)P(Z = z) = (p_a + p_c)r + p_a(1 - r) = p_a + rp_c$. Plugging these expressions into formula (A.6), we have

$$P(Z = 1|R = 1) = \frac{(p_a+p_c)r}{p_a+rp_c}, \quad P(Z = 0|R = 1) = \frac{p_a(1-r)}{p_a+rp_c} \quad (A.7)$$

Step 3: identify τ^R.

Plugging (A.5) and (A.7) into (A.2), we obtain the marginal probability of always-recruited and compliant-recruited in the recruited sample to be

$$P(S = c|R = 1) = \frac{rp_c}{rp_c + p_a}, \quad P(S = a|R = 1) = 1 - \frac{rp_c}{rp_c + p_a} \quad (A.8)$$

Plugging (A.8) into (A.1), we prove

$$\tau^R = \frac{rp_c}{rp_c+p_a}\tau_c + \left(1 - \frac{rp_c}{rp_c+p_a}\right)\tau_a. \qquad \blacksquare$$

B. Randomization probability to ensure balanced samples between arms.

If we want to ensure the sample size of the recruited subjects are similar between two arms, just setting $P(Z = 1|R = 1) = P(Z = 0|R = 1)$. Plugging the expressions in (A.7) into the equation we obtain: $(p_a + p_c)r = p_a(1 - r)$. Solving r gives $r = p_a/(2p_a + p_c)$.