Web Appendix for "Propensity score weighting for covariate adjustment in randomized clinical trials" by Zeng et al.

APPENDIX A: PROOFS OF THE PROPOSITIONS IN SECTION 3.1

We proceed under a set of standard regularity conditions such as the expectations $E(Y_i|X_i, Z_i)$, $E(Y_i^2|X_i, Z_i)$ are finite and well defined. We assume that the treatment Z is randomly assigned to patients, where $\Pr(Z_i = 1|X_i, Y_i(1), Y_i(0)) = \Pr(Z_i = 1) = r$, and $0 < r < 1$ is the randomization probability. We allow the joint distribution $\Pr(Z_1, Z_2, \cdots, Z_N)$ to be flexible as long as $\Pr(Z_i = 1) = r$ is fixed. This includes the case where we assign each unit treatment independently with probability r (N_1 and N_0 are random variables) or the case where we assign a fixed proportion into the treatment group (N_1 and N_0 are fixed). In the former case, we assume r is bounded away from 0 and 1 so that $\Pr(N_1 = 0)$ and $\Pr(N_0 = 0)$ are negligible (otherwise the weighting estimator may be undefined).

Proof for Proposition 1(a). Suppose the propensity score model $e_i = e(X_i; \theta)$ is a smooth function of θ, and the estimated parameter $\hat{\theta}$ is obtained by maximum likelihood, we derive the score function $S_{\theta,i}$ for each observation i, namely the first order derivative of the log likelihood with respect to θ,

$$S_{\theta,i} = \frac{\partial}{\partial\theta} l_i(\theta) = \frac{\partial}{\partial\theta}\left\{Z_i \log e(X_i;\theta) + (1 - Z_i)\log(1 - e(X_i;\theta))\right\} = \frac{Z_i - e(X_i;\theta)}{e(X_i;\theta)(1 - e(X_i;\theta))}\frac{\partial e(X_i;\theta)}{\partial\theta},$$

where $\frac{\partial e(X_i;\theta)}{\partial\theta}$ is the derivative evaluated at θ. As the true probability of being treated is a constant r and the logistic model is always correctly specified as long as it includes an intercept, there exists θ^* such that $e(X_i; \theta^*) = r$. When $\theta = \theta^*$, the score function is,

$$S_{\theta^*,i} = \frac{Z_i - r}{r(1-r)}\frac{\partial e(X_i;\theta^*)}{\partial\theta}.$$

Let $I_{\theta\theta}$ be the information matrix evaluated at θ, whose exact form is,

$$I_{\theta\theta} = E\left\{\frac{\partial}{\partial\theta} l_i(\theta)\frac{\partial}{\partial\theta} l_i(\theta)^T\right\} = E\left\{\frac{(Z_i - e(X_i;\theta))^2}{(e(X_i;\theta)(1 - e(X_i;\theta)))^2}\frac{\partial e(X_i;\theta)}{\partial\theta}\frac{\partial e(X_i;\theta)}{\partial\theta}^T\right\}.$$

When $\theta = \theta^*$,

$$I_{\theta^*\theta^*} = \frac{1}{r(1-r)} E\left\{\frac{\partial e(X_i;\theta^*)}{\partial\theta}\frac{\partial e(X_i;\theta^*)}{\partial\theta}^T\right\}.$$

Applying the Cramer-Rao theorem, assume the propensity score model $e(X_i; \theta)$ satisfies certain regularity conditions[1], the Taylor expansion $\hat{\theta}$ at true value is,

$$\sqrt{N}(\hat{\theta} - \theta^*) = I^{-1}_{\theta^*\theta^*}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} S_{\theta^*,i} + o_p(1),$$

By the Weak Law of Large Numbers (WLLN), we can establish the consistency of $\hat{\theta}$,

$$\hat{\theta} - \theta^* \xrightarrow{p} I^{-1}_{\theta^*\theta^*} E(S_{\theta^*,i}) = I^{-1}_{\theta^*\theta^*}\frac{E(Z_i - r)}{r(1-r)} E\left\{\frac{\partial e(X_i;\theta^*)}{\partial\theta}\right\} = 0.$$

With the consistency of $\hat{\theta}$, we also have,

$$\frac{1}{N}\sum_{i=1}^{N} Z_i(1 - e(X_i;\hat{\theta})) \xrightarrow{p} r(1-r), \frac{1}{N}\sum_{i=1}^{N}(1 - Z_i)e(X_i;\hat{\theta}) \xrightarrow{p} r(1-r).$$

Next, we investigate the influence function of $\hat{\mu}_1^{\text{ow}} - \hat{\mu}_0^{\text{ow}}$,

$$\begin{aligned}\sqrt{N}(\hat{\mu}_1^{\text{ow}} - \hat{\mu}_0^{\text{ow}}) =& \sqrt{N}\left(\frac{\sum_{i=1}^{N} Z_i Y_i(1 - e(X_i;\hat{\theta}))}{\sum_{i=1}^{N} Z_i(1 - e(X_i;\hat{\theta}))} - \frac{\sum_{i=1}^{N}(1 - Z_i)Y_i e(X_i;\hat{\theta})}{\sum_{i=1}^{N}(1 - Z_i)e(X_i;\hat{\theta})}\right),\\ =& \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\{\frac{Z_i Y_i(1 - e(X_i;\hat{\theta}))}{r(1-r)} - \frac{(1 - Z_i)Y_i e(X_i;\hat{\theta})}{r(1-r)}\right\} + o_p(1).\end{aligned}$$

We perform the Taylor expansion at the true value θ^*,

$$\begin{aligned}
\sqrt{N}(\hat{\mu}_1^{\text{ow}} - \hat{\mu}_0^{\text{ow}}) =& \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \frac{Z_i Y_i (1 - e(X_i;\hat{\theta}))e(X_i;\hat{\theta})}{e(X_i;\hat{\theta})r(1-r)} - \frac{(1-Z_i)Y_i(1-e(X_i;\hat{\theta}))e(X_i;\hat{\theta})}{(1-e(X_i;\hat{\theta}))r(1-r)} \right\} + o_p(1), \\
=& \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \frac{Z_i Y_i (1 - e(X_i;\theta^*))e(X_i;\theta^*)}{e(X_i;\theta^*)r(1-r)} - \frac{(1-Z_i)Y_i(1-e(X_i;\theta^*))e(X_i;\theta^*)}{(1-e(X_i;\theta^*))r(1-r)} \right\} - \\
& \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \frac{Z_i Y_i (1 - e(X_i;\theta^*))e(X_i;\theta^*)}{e(X_i;\theta^*)r(1-r)} - \frac{(1-Z_i)Y_i(1-e(X_i;\theta^*))e(X_i;\theta^*)}{(1-e(X_i;\theta^*))r(1-r)} \right\} S_{\theta^*,i}^T(\hat{\theta} - \theta^*) + o_p(1), \\
=& \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} - \frac{1}{N} \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i}^T \right] \sqrt{N}(\hat{\theta} - \theta^*) + o_p(1), \\
=& \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} - E \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i}^T \right] I_{\theta^*\theta^*}^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} S_{\theta^*,i} + o_p(1).
\end{aligned}$$

After plugging in the value of $S_{\theta^*,i}$ and $I_{\theta^*\theta^*}$, we can show that,

$$\begin{aligned}
\hat{\mu}_1^{\text{ow}} - \hat{\mu}_0^{\text{ow}} &= \frac{1}{N} \sum_{i=1}^{N} \left[\frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} - \frac{Z_i - r}{r(1-r)} \left\{ (1-r)g_1(X_i) + r g_0(X_i) \right\} \right] + o_p(N^{-1/2}) \\
g_1(X_i) &= E \left[Y_i \frac{\partial e(X_i;\theta^*)}{\partial \theta} \middle| Z_i = 1 \right]^T E \left\{ \frac{\partial e(X_i;\theta^*)}{\partial \theta} \frac{\partial e(X_i;\theta^*)}{\partial \theta}^T \right\}^{-1} \frac{\partial e(X_i;\theta^*)}{\partial \theta}, \\
g_0(X_i) &= E \left[Y_i \frac{\partial e(X_i;\theta^*)}{\partial \theta} \middle| Z_i = 0 \right]^T E \left\{ \frac{\partial e(X_i;\theta^*)}{\partial \theta} \frac{\partial e(X_i;\theta^*)}{\partial \theta}^T \right\}^{-1} \frac{\partial e(X_i;\theta^*)}{\partial \theta}.
\end{aligned}$$

Therefore, $\hat{\tau}^{\text{ow}}$ belongs to the augmented IPW estimator class $\mathcal{I}$ in the main text, which completes the proof of Proportion 1 (a).

Proof for Proposition 1(b): First, we build the relationship between the asymptotic variance of $\hat{\tau}^{\text{ow}}$ with the correpsonding information matrix $I_{\theta^*\theta^*}$ and score function $S_{\theta^*,i}$ evaluated at true value. Based on the results in Proposition 1(a), the asymptotic variance of $\hat{\tau}^{\text{ow}}$ depends on the following terms:

$$\begin{aligned}
\lim_{N\to\infty} N \text{Var}(\hat{\tau}^{\text{ow}}) =& \text{Var} \left(\frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} - E \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i}^T \right] I_{\theta^*\theta^*}^{-1} S_{\theta^*,i} \right), \\
=& \text{Var} \left(\frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right) + \text{Var} \left(E \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i}^T \right] I_{\theta^*\theta^*}^{-1} S_{\theta^*,i} \right), \\
& -2\text{Cov} \left(\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\}, E \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i}^T \right] I_{\theta^*\theta^*}^{-1} S_{\theta^*,i} \right).
\end{aligned}$$

Notice the facts that

$$\begin{aligned}
& E \left(\frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right) = 0, E(S_{\theta^*,i}) = 0, \\
E(S_{\theta^*,i} S_{\theta^*,i}^T) = E \left\{ \frac{(Z_i - r)^2}{r^2(1-r)^2} \right\} & E \left\{ \frac{\partial e(X_i;\theta^*)}{\partial \theta} \frac{\partial e(X_i;\theta^*)}{\partial \theta}^T \right\} = \frac{1}{(1-r)r} E \left\{ \frac{\partial e(X_i;\theta^*)}{\partial \theta} \frac{\partial e(X_i;\theta^*)}{\partial \theta}^T \right\} = I_{\theta^*\theta^*},
\end{aligned}$$

we have,

$$\begin{aligned}
\text{Var} \left(E \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i}^T \right] I_{\theta^*\theta^*}^{-1} S_{\theta^*,i} \right) =& E \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i}^T \right] I_{\theta^*\theta^*}^{-1} E \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i} \right], \\
=& \text{Cov} \left(\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\}, E \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i}^T \right] I_{\theta^*\theta^*}^{-1} S_{\theta^*,i} \right).
\end{aligned}$$

We can further reduce the asymptotic variance to,

$$\lim_{N\to\infty} N \text{Var}(\hat{\tau}^{\text{ow}}) = \text{Var} \left(\frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right) - \text{Var} \left(E \left[\left\{ \frac{Z_i Y_i}{r} - \frac{(1-Z_i)Y_i}{1-r} \right\} S_{\theta^*,i}^T \right] I_{\theta^*\theta^*}^{-1} S_{\theta^*,i} \right).$$

Recall that X^1 and X^2 denote two nested sets of covariates with $X^2 = (X^1, X^{*1})$, and $e(X_i^1;\theta_1)$, $e(X_i^2;\theta_2)$ are the nested smooth parametric propensity score models. Suppose $\hat{\tau}_1^{\text{ow}}$ and $\hat{\tau}_2^{\text{ow}}$ are two OW estimators derived from the fitted propensity score

$e(X_i^1;\hat\theta_1)$ and $e(X_i^1;\hat\theta_2)$ respectively. Denote the true value of the nested propensity score models as θ^{*1},θ^{*2}, the score functions at true value as $S_{\theta^{*1},i}$,$S_{\theta^{*2},i}$ and the information matrix as $I_{\theta^{*1},\theta^{*1}}$ and $I_{\theta^{*2},\theta^{*2}}$. To prove $\lim_{N\to\infty} N\mathrm{Var}(\hat\tau_1^{\mathrm{ow}}) \geq \lim_{N\to\infty} N\mathrm{Var}(\hat\tau_2^{\mathrm{ow}})$, it is equivalent to establish the following inequality,

$$\mathrm{Var}\left(E\left[\left\{\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}\right\}S^T_{\theta^{*2},i}\right]I^{-1}_{\theta^{*2}\theta^{*2}}S_{\theta^{*2},i}\right) \geq \mathrm{Var}\left(E\left[\left\{\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}\right\}S^T_{\theta^{*1},i}\right]I^{-1}_{\theta^{*1}\theta^{*1}}S_{\theta^{*1},i}\right).$$

Using the equivalent expression, this inequality becomes,

$$E\left[\left\{\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}\right\}S^T_{\theta^{*2},i}\right]I^{-1}_{\theta^{*2}\theta^{*2}}E\left[\left\{\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}\right\}S_{\theta^{*2},i}\right] \geq$$
$$E\left[\left\{\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}\right\}S^T_{\theta^{*1},i}\right]I^{-1}_{\theta^{*1}\theta^{*1}}E\left[\left\{\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}\right\}S_{\theta^{*1},i}\right].$$

Additionally, as the two models are nested,

$$I_{\theta^{*2}\theta^{*2}} = \begin{bmatrix} I_{\theta^{*1}\theta^{*1}} & I^{12}_{\theta^{*2}\theta^{*2}} \\ I^{21}_{\theta^{*2}\theta^{*2}} & I^{22}_{\theta^{*2}\theta^{*2}} \end{bmatrix} \triangleq \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}, E\left[\left\{\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}\right\}S_{\theta^{*2},i}\right] = E\begin{bmatrix} \left\{\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}\right\}S_{\theta^{*1},i} \\ \left\{\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}\right\}S^2_{\theta^{*1},i} \end{bmatrix} \triangleq \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}.$$

The inverse of the information matrix for the larger model is

$$I^{-1}_{\theta^{*2}\theta^{*2}} = \begin{bmatrix} I_{11}^{-1} + I_{11}^{-1}I_{12}(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1}I_{21}I_{11}^{-1} & -I_{11}^{-1}I_{12}(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1} \\ -(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1}I_{21}I_{11}^{-1} & (I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1} \end{bmatrix}.$$

Hence we can calculate the difference of asymptotic variance,

$$\begin{aligned}\lim_{N\to\infty} N\mathrm{Var}(\hat\tau_1^{\mathrm{ow}}) - \lim_{N\to\infty} N\mathrm{Var}(\hat\tau_2^{\mathrm{ow}}) =& U_1^T\{I_{11}^{-1}I_{12}(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1}I_{21}I_{11}^{-1}\}U_1 - U_1^TI_{11}^{-1}I_{12}(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1}U_2 \\ & -U_2^T(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1}I_{21}I_{11}^{-1}U_1 + U_2^T(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1}U_2^T, \\ =& (I_{21}I_{11}^{-1}U_1-U_2)^T(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1}(I_{21}I_{11}^{-1}U_1-U_2) \geq 0.\end{aligned}$$

The last inequality follows from the fact that $(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1}$ is positive definite. Hence, we have proved the asymptotic variance of the $\hat\tau_2^{\mathrm{ow}}$ is no greater than the OW estimator $\hat\tau_1^{\mathrm{ow}}$ with fewer covariates, which completes the proof of Proposition 1(b).

Proof for Proposition 1(c): When we are using logistic regression to estimate the propensity score, we have $\frac{\partial e(X_i;\theta^*)}{\partial\theta} = r(1-r)\tilde X_i, \tilde X_i = (1, X_i^T)^T$. Plugging this quantity into the g_1, g_0, we have,

$$\begin{aligned} g_1(X_i) &= E(Y_i\tilde X_i|Z_i=1)^TE(\tilde X_i\tilde X_i^T)^{-1}\tilde X_i, \\ &= E(Y_i\tilde X_i|Z_i=1)^TE(\tilde X_i\tilde X_i^T|Z_i=1)^{-1}\tilde X_i, \\ g_0(X_i) &= E(Y_i\tilde X_i|Z_i=0)^TE(\tilde X_i\tilde X_i^T|Z_i=0)^{-1}\tilde X_i, \end{aligned}$$

where g_0 and g_1 correspond to linear projection of Y_i into the space of X_i (including a constant) in two arms. If the true outcome surface $E(Y_i|X_i,Z_i=1)$ and $E(Y_i|X_i,Z_i=0)$ are indeed linear functions of X_i, then the $g_1(X_i)=E(Y_i|X_i,Z_i=1)$,$g_0(X)=E(Y_i|X_i,Z_i=0)$, $\hat\tau^{\mathrm{ow}} = \hat\mu_1{}^{\mathrm{ow}} - \hat\mu_0{}^{\mathrm{ow}}$ is semiparametric efficient. As such, we complete the proof of Proposition 1(c).

Proof for Proposition 2: Since we require $h(x)$ to be a function of the propensity score, we denote the tilting function and the resulting balancing weights as $h(X_i;\theta)$, $w_1(X_i;\theta)$, $w_0(X_i;\theta)$ corresponding to each observation i. Also, we make the following assumptions:

(i) (Nonzero tilting function) There exists $\varepsilon > 0$ such that $P\{h(X_i;\theta^*) > \varepsilon\} = 1$.

(ii) (Smoothness) the first and second order derivatives of balancing weights with respect to the propensity score $\frac{d}{de}w_1(X_i;\theta)\}$, $\frac{d}{de}w_0(X_i;\theta)$,$\frac{d^2}{de^2}w_1(X_i;\theta)\}$, $\frac{d^2}{de^2}w_0(X_i;\theta)$ exists and are continuous in e.

(iii) (Bounded derivative in the neighborhood of θ^*) For the true value θ^*, there exists $c > 0$ and $M_1 > 0, M_2 > 0$ such that

$$\left|\frac{d}{de}w_0(X_i;\theta^*)\right| \leq M_1, \left|\frac{d}{de}w_0(X_i;\theta^*)\right| \leq M_1$$
$$\left|\frac{d^2}{de^2}w_0(X_i;\theta)\right| \leq M_2, \left|\frac{d^2}{de^2}w_1(X_i;\theta)\right| \leq M_2,$$

almost surely for θ in the neighborhood of θ^*, i.e. $\theta \in \{\theta|\,||\theta-\theta^*||_1 \leq c\}$.

We investigate the influence function of $\hat{\tau}^h$ for a given $h(x)$ and do Taylor expansion at the true value θ^*,

$$\begin{aligned}
\sqrt{N}(\hat{\mu}_1^h - \hat{\mu}_0^h) =& \sqrt{N}\left(\frac{\sum_{i=1}^N Z_i Y_i w_1(X_i;\hat{\theta})}{\sum_{i=1}^N Z_i w_1(X_i;\hat{\theta})} - \frac{\sum_{i=1}^N (1-Z_i) Y_i w_0(X_i;\hat{\theta})}{\sum_{i=1}^N (1-Z_i) w_0(X_i;\hat{\theta})}\right),\\
=& \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^N Z_i Y_i w_1(X_i;\hat{\theta})}{E\{h(X_i;\theta^*)\}} - \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^N (1-Z_i) Y_i w_0(X_i;\hat{\theta})}{E\{h(X_i;\theta^*)\}} + o_p(1),\\
=& \frac{1}{\sqrt{N}}\sum_{i=1}^N \left\{ \frac{Z_i Y_i w_1(X_i;\theta^*)}{E\{h(X_i;\theta^*)\}} - \frac{(1-Z_i) Y_i w_0(X_i;\theta^*)}{E\{h(X_i;\theta^*)\}} \right.\\
&+ \frac{\left\{ Z_i Y_i \frac{d}{de} w_1(X_i;\theta^*) - (1-Z_i) Y_i \frac{d}{de} w_0(X_i;\theta^*)\right\} \frac{\partial e(X_i;\theta^*)}{\partial \theta}^T (\hat{\theta}-\theta^*)}{E\{h(X_i;\theta^*)\}}\\
&+ \left.\frac{\left\{ Z_i Y_i \left[\frac{d^2}{de^2} w_1(X_i;\tilde{\theta}) + \frac{d}{de} w_1(X_i;\tilde{\theta})\right] - (1-Z_i) Y_i \left[\frac{d^2}{de^2} w_0(X_i;\tilde{\theta}) + \frac{d}{de} w_0(X_i;\tilde{\theta})\right]\right\} (\hat{\theta}-\theta^*)^T \frac{\partial^2 e(X_i;\tilde{\theta})}{\partial \theta^2} (\hat{\theta}-\theta^*)}{E\{h(X_i;\theta^*)\}} \right\}\\
&+ o_p(1),
\end{aligned}$$

where $\tilde{\theta}$ lies in the line between θ^* and $\hat{\theta}$, such that $\tilde{\theta} = \theta^* + t(\hat{\theta} - \theta^*)$, $t \in (0, 1)$ (Taylor expansion with Lagrange remainder term). To see that the third term converges to zero in probability, we have $\sqrt{N}(\hat{\theta} - \theta^*)$ is asymptotic normal distributed with Cramer-Rao theorem and the asymptotic covariance is proportional to $E\left\{\frac{\partial^2 e(X_i;\theta^*)}{\partial \theta^2}\right\}^{-1}$, which means $N(\hat{\theta}-\theta^*)^T E\left\{\frac{\partial^2 e(X_i;\theta^*)}{\partial \theta^2}\right\}(\hat{\theta}-\theta^*)$ is tight, or equivalently

$$P\left\{ N(\hat{\theta}-\theta^*)^T E\left\{\frac{\partial^2 e(X_i;\theta^*)}{\partial \theta^2}\right\}(\hat{\theta}-\theta^*) < \infty \right\} = 1.$$

Secondly, as $\hat{\theta} \xrightarrow{p} \theta^*$, $\tilde{\theta} \xrightarrow{p} \theta^*$, when N is sufficiently large, $||\tilde{\theta} - \theta^*||_1 \le c$, the first and second order derivative is bounded almost surely, such that

$$\left|\frac{d^2}{de^2} w_1(X_i;\tilde{\theta}) + \frac{d}{de} w_1(X_i;\tilde{\theta})\right| \le M_1 + M_2, \left|\frac{d^2}{de^2} w_0(X_i;\tilde{\theta}) + \frac{d}{de} w_0(X_i;\tilde{\theta})\right| \le M_1 + M_2.$$

Therefore, by the WLLN,

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \left\{ Z_i Y_i \left[\frac{d^2}{de^2} w_1(X_i;\tilde{\theta}) + \frac{d}{de} w_1(X_i;\tilde{\theta})\right] - (1-Z_i) Y_i \left[\frac{d^2}{de^2} w_0(X_i;\tilde{\theta}) + \frac{d}{de} w_0(X_i;\tilde{\theta})\right]\right\}\\
&\le (M_1 + M_2)\frac{1}{N}\sum_{i=1}^N |Z_i Y_i| + |(1-Z_i) Y_i)| \xrightarrow{p} E\{|Z_i Y_i| + |(1-Z_i) Y_i)|\} < \infty.
\end{aligned}$$

Also, as $\tilde{\theta} \xrightarrow{p} \theta^*$, and we assume $e(X_i;\theta)$ is smooth (so that $\frac{\partial^2 e(X_i;\tilde{\theta})}{\partial \theta^2}$ is continuous),

$$\frac{1}{N}\sum_{i=1}^N \frac{\partial^2 e(X_i;\tilde{\theta})}{\partial \theta^2} \xrightarrow{p} E\left\{\frac{\partial^2 e(X_i;\theta^*)}{\partial \theta^2}\right\}.$$

As such, we can conclude that the third term converges to zero in probability,

$$\begin{aligned}
&\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{\left\{ Z_i Y_i \left[\frac{d^2}{de^2} w_1(X_i;\tilde{\theta}) + \frac{d}{de} w_1(X_i;\tilde{\theta})\right] - (1-Z_i) Y_i \left[\frac{d^2}{de^2} w_0(X_i;\tilde{\theta}) + \frac{d}{de} w_0(X_i;\tilde{\theta})\right]\right\} (\hat{\theta}-\theta^*)^T \frac{\partial^2 e(X_i;\tilde{\theta})}{\partial \theta^2} (\hat{\theta}-\theta^*)}{E\{h(X_i;\theta^*)\}}\\
=& O_p\left(\frac{1}{\sqrt{N}} \frac{E\{|Z_i Y_i| + |(1-Z_i) Y_i)|\} N(\hat{\theta}-\theta^*)^T E\left\{\frac{\partial^2 e(X_i;\theta^*)}{\partial \theta^2}\right\}(\hat{\theta}-\theta^*)}{E\{h(X_i;\theta^*)\}}\right) \xrightarrow{p} 0.
\end{aligned}$$

Hence, we have,

$$\sqrt{N}(\hat{\mu}_1^h - \hat{\mu}_0^h) = \frac{1}{\sqrt{N}}\sum_{i=1}^N \left\{ \frac{Z_i Y_i w_1(X_i;\theta^*)}{E\{h(X_i;\theta^*)\}} - \frac{(1-Z_i) Y_i w_0(X_i;\theta^*)}{E\{h(X_i;\theta^*)\}} \right.$$

$$
+\frac{\left\{Z_iY_i\frac{d}{de}w_1(X_i;\theta^*)-(1-Z_i)Y_i\frac{d}{de}w_0(X_i;\theta^*)\right\}\frac{\partial e(X_i;\theta^*)}{\partial\theta}(\hat{\theta}-\theta^*)}{E\{h(X_i;\theta^*)\}}\Bigg\}+o_p(1),
$$

$$
=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\Bigg\{\frac{Z_iY_ih(X_i;\theta^*)/r}{E\{h(X_i;\theta^*)\}}-\frac{(1-Z_i)Y_ih(X_i;\theta^*)/(1-r)}{E\{h(X_i;\theta^*)\}}
$$

$$
+\frac{E\left[\left\{Z_iY_i\frac{d}{de}w_1(X_i;\theta^*)-(1-Z_i)Y_i\frac{d}{de}w_0(X_i;\theta^*)\right\}\frac{\partial e(X_i;\theta^*)}{\partial\theta}^T\right]}{E\{h(X_i;\theta^*)\}}I_{\theta^*\theta^*}^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}S_{\theta^*,i}\Bigg\}+o_p(1).
$$

Since $h(X_i;\theta)$ is a function of propensity score, $h(X_i;\theta^*)$ is a function of r, which means $E\{h(X_i;\theta^*)\}=h(X_i;\theta^*)$. Applying this property and plugging in the value of $S_{\theta^*,i}, I_{\theta^*\theta^*}$, we have,

$$
\hat{\mu}_1^h-\hat{\mu}_0^h=\frac{1}{N}\sum_{i=1}^{N}\left[\frac{Z_iY_i}{r}-\frac{(1-Z_i)Y_i}{1-r}-\frac{Z_i-r}{r(1-r)}\left\{(1-r)g_1^h(X_i)+rg_0^h(X_i)\right\}\right]+o_p(N^{-1/2}),
$$

$$
g_1^h(X_i)=-\frac{r}{h(X_i;\theta^*)}E\left\{Z_iY_i\frac{d}{de}w_1(X_i;\theta^*)\right\}\frac{\partial e(X_i;\theta^*)}{\partial\theta}^T E\left\{\frac{\partial e(X_i;\theta^*)}{\partial\theta}\frac{\partial e(X_i;\theta^*)}{\partial\theta}^T\right\}^{-1}\frac{\partial e(X_i;\theta^*)}{\partial\theta},
$$

$$
g_0^h(X_i)=\frac{1-r}{h(X_i;\theta^*)}E\left\{(1-Z_i)Y_i\frac{d}{de}w_0(X_i;\theta^*)\right\}\frac{\partial e(X_i;\theta^*)}{\partial\theta}^T E\left\{\frac{\partial e(X_i;\theta^*)}{\partial\theta}\frac{\partial e(X_i;\theta^*)}{\partial\theta}^T\right\}^{-1}\frac{\partial e(X_i;\theta^*)}{\partial\theta},
$$

which completes the proof of Proposition 2.

References

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APPENDIX B: DERIVATION OF THE ASYMPTOTIC VARIANCE AND ITS CONSISTENT ESTIMATOR IN SECTION 3.3

Asymptotic variance derivation. As we have shown in the main text (Section 3.3), the asymptotic variance of $\hat{\tau}^{\text{OW}}$ depends on the elements in the sandwich matrix $A^{-1}BA^{-T}$, where $A = -E(\partial U_i/\partial \lambda), B = E(U_i U_i^T)$ evaluated at the true parameter value (μ_1, μ_0, θ^*). The exact form of the matrices A and B are as follows:

$$A = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, A^{-1} = \begin{bmatrix} a_{11}^{-1} & 0 & -a_{11}^{-1}a_{13}a_{33}^{-1}, \\ 0 & a_{22}^{-1} & -a_{22}^{-1}a_{23}a_{33}^{-1} \\ 0 & 0 & a_{33}^{-1} \end{bmatrix}, B = \begin{bmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ b_{13}^T & b_{23}^T & b_{33} \end{bmatrix},$$

$$a_{11} = E\{Z_i(1-e_i)\}, a_{13} = E\{\tilde{X}_i^T(Y_i - \mu_1)Z_i e_i(1-e_i)\}, a_{22} = E\{(1-Z_i)e_i\},$$

$$a_{23} = -E(\tilde{X}_i^T(Y_i - \mu_0)(1-Z_i)e_i(1-e_i), a_{33} = E(e_i(1-e_i)\tilde{X}\tilde{X}^T],$$

$$b_{11} = E\{(Y_i - \mu_1)^2 Z_i(1-e_i)^2\}, b_{13} = E\{\tilde{X}_i^T(Y_i - \mu_1)Z_i(Z_i - e_i)(1-e_i), b_{23} = E\{\tilde{X}_i^T(Y_i - \mu_0)(1-Z_i)(Z_i - e_i)e_i\},$$

$$b_{22} = E\{(Y_i - \mu_0)^2(1-Z_i)e_i^2\}, b_{33} = E\{(Z_i - e_i)^2\tilde{X}_i\tilde{X}_i^T\}.$$

After multiplying $A^{-1}BA^{-T}$ and extracting the upper left 2×2 matrix, we have,

$$\Sigma_{11} = [A^{-1}BA^{-T}]_{1,1} = \frac{1}{a_{11}^{-2}}(b_{11} - 2a_{13}a_{33}^{-1}b_{13}^T + a_{13}a_{33}^{-1}b_{33}a_{33}^{-1}a_{13}^T),$$

$$\Sigma_{22} = [A^{-1}BA^{-T}]_{2,2} = \frac{1}{a_{22}^{-2}}(b_{22} - 2a_{23}a_{33}^{-1}b_{23}^T + a_{23}a_{33}^{-1}b_{33}a_{33}^{-1}a_{23}^T),$$

$$\Sigma_{12} = \Sigma_{21} = [A^{-1}BA^{-T}]_{1,2} = \frac{1}{a_{11}a_{22}}(-a_{13}a_{33}^{-1}b_{23}^T - a_{23}a_{33}^{-1}b_{13}^T + a_{13}a_{33}^{-1}b_{33}a_{33}^{-1}a_{23}^T).$$

With the delta method, we can express the asymptotic variance for $\hat{\tau}_{\text{RD}}^{\text{OW}}, \hat{\tau}_{\text{RR}}^{\text{OW}}, \hat{\tau}_{\text{OR}}^{\text{OW}}$,

$$\text{Var}(\hat{\tau}_{\text{RD}}^{\text{OW}}) = \frac{1}{N}\left(\Sigma_{11} + \Sigma_{22} - 2\Sigma_{12}\right),$$

$$\text{Var}(\hat{\tau}_{\text{RR}}^{\text{OW}}) = \frac{1}{N}\left(\frac{\Sigma_{11}}{\mu_1^2} + \frac{\Sigma_{22}}{\mu_0^2} - \frac{2\Sigma_{12}}{\mu_1\mu_0}\right),$$

$$\text{Var}(\hat{\tau}_{\text{OR}}^{\text{OW}}) = \frac{1}{N}\left\{\frac{\Sigma_{11}}{\mu_1^2(1-\mu_1)^2} + \frac{\Sigma_{22}}{\mu_0^2(1-\mu_0)^2} - \frac{2\Sigma_{12}}{\mu_1(1-\mu_1)\mu_0(1-\mu_0)}\right\}.$$

Specifically, we write out the exact form of large sample variance for the estimator on additive scale after exploiting the fact that $E(Z_i) = E(e_i) = r$,

$$NVar(\hat{\tau}^{\text{OW}}) \rightarrow \frac{\text{Var}(Y_i|Z_i = 1)}{r} + \frac{\text{Var}(Y_i|Z_i = 1)}{1-r} - \frac{\{rm_1 + (1-r)m_0\}E(\tilde{X}_i\tilde{X}_i^T)^{-1}\{(2-3r)m_1 + (3r-1)m_0\}}{r(1-r)},$$

where $m_1 = E(\tilde{X}_i(Y_i - \mu_1)|Z_i = 1), m_0 = E(\tilde{X}_i(Y_i - \mu_1)|Z_i = 1)$

Connection to R-squared: When $r = 0.5$, the large sample variance of $\hat{\tau}^{\text{OW}}$ is,

$$\begin{aligned} N\text{Var}(\hat{\tau}^{\text{OW}}) \rightarrow & 2\left\{\text{Var}(Y_i|Z_i = 1) + \text{Var}(Y_i|Z_i = 0)\right\} - 4(\frac{1}{2}m_1 + \frac{1}{2}m_0)E(\tilde{X}\tilde{X}^T)^{-1}(\frac{1}{2}m_1 + \frac{1}{2}m_0), \\ =& 2\left\{\text{Var}(Y_i|Z_i = 1) + \text{Var}(Y_i|Z_i = 0)\right\} - 4E(\tilde{X}_i\tilde{Y}_i)E(\tilde{X}_i\tilde{X}_i^T)^{-1}E(\tilde{X}_i\tilde{Y}_i), \\ =& 2\left\{\text{Var}(Y_i|Z_i = 1) + \text{Var}(Y_i|Z_i = 0)\right\} - 4R^2_{\tilde{Y}\sim X}\text{Var}(\tilde{Y}_i), \\ =& 2\left\{\text{Var}(Y_i|Z_i = 1) + \text{Var}(Y_i|Z_i = 0)\right\} - 2R^2_{\tilde{Y}\sim X}\left\{\text{Var}(Y_i|Z_i = 1) + \text{Var}(Y_i|Z_i = 0)\right\}, \\ =& 4(1 - R^2_{\tilde{Y}\sim X})\text{Var}(\tilde{Y}_i), \\ =& \lim_{N\rightarrow\infty}(1 - R^2_{\tilde{Y}\sim X})N\text{Var}(\hat{\tau}^{\text{UNADJ}}). \end{aligned}$$

where $\tilde{Y}_i = Z_i(Y_i - \mu_1) + (1 - Z_i)(Y_i - \mu_0)$. In the derivation, we use the fact that,

$$\text{Var}(\tilde{Y}_i) = E(\tilde{Y}_i^2) - E(\tilde{Y}_i)^2 = \frac{1}{2}E\left((Y_i - \mu_1)^2|Z_i = 1\right) + \frac{1}{2}E\left((Y_i - \mu_1)^2|Z_i = 1\right) = \frac{1}{2}\left\{\text{Var}(Y_i|Z_i = 1) + \text{Var}(Y_i|Z_i = 0)\right\}.$$

The efficiency gain is regardless of whether our model is correctly specified or not. Additionally, if we augment the covariate space from $\tilde{X}_i$ to X_i^*, the $R^2_{\tilde{Y}\sim X}$ is non-decreasing with $R^2_{\tilde{Y}\sim X} \leq R^2_{\tilde{Y}\sim X^*}$. Therefore, the asymptotic variance of OW estimator with additional covariates decreases, $\text{Var}(\hat{\tau}^{\text{OW}*}) \leq \text{Var}(\hat{\tau}^{\text{OW}})$. This provides a heuristic justification of Proposition 1(b) when $r = 0.5$.

Consistent variance estimator: We obtain the empirical estimator for the asymptotic variance by plugging in the finite sample estimate for the elements in the sandwich matrix $A^{-1}BA^{-T}$,

$$\hat{\Sigma}_{11} = \frac{1}{\hat{a}_{11}^2}(\hat{b}_{11} - 2\hat{a}_{13}\hat{a}_{33}^{-1}\hat{b}_{13}^T + \hat{a}_{13}\hat{a}_{33}^{-1}\hat{a}_{13}^T),$$
$$\hat{\Sigma}_{22} = \frac{1}{\hat{a}_{11}^2}(\hat{b}_{22} - 2\hat{a}_{23}\hat{a}_{33}^{-1}\hat{b}_{23}^T + \hat{a}_{23}\hat{a}_{33}^{-1}\hat{a}_{23}^T),$$
$$\hat{\Sigma}_{12} = -\frac{1}{\hat{a}_{11}^2}(\hat{a}_{13}\hat{a}_{33}^{-1}\hat{b}_{23}^T + \hat{a}_{23}\hat{a}_{33}^{-1}\hat{b}_{13}^T - \hat{a}_{13}\hat{a}_{33}^{-1}\hat{a}_{23}^T),$$

$$\hat{a}_{11} = \hat{a}_{22} = \frac{1}{N}\sum_{i=1}^N\{\hat{e}_i(1-\hat{e}_i)\}, \qquad \hat{a}_{33} = \hat{b}_{33} = \frac{1}{N}\sum_{i=1}^N\{\hat{e}_i(1-\hat{e}_i)\tilde{X}_i^T\tilde{X}_i\},$$
$$\hat{a}_{13} = \frac{1}{N_1}\sum_{i=1}^N Z_i\hat{e}_i^2(1-\hat{e}_i)(Y_i-\hat{\mu}_1)^2\tilde{X}_i, \qquad \hat{a}_{23} = \frac{1}{N_0}\sum_{i=1}^N(1-Z_i)\hat{e}_i(1-\hat{e}_i)^2(Y_i-\hat{\mu}_0)^2\tilde{X}_i,$$
$$\hat{b}_{11} = \frac{1}{N_1}\sum_{i=1}^N Z_i\hat{e}_i(1-\hat{e}_i)^2(Y_i-\hat{\mu}_1)^2, \qquad \hat{b}_{22} = \frac{1}{N_0}\sum_{i=1}^N(1-Z_i)\hat{e}_i^2(1-\hat{e}_i)(Y_i-\hat{\mu}_0)^2,$$
$$\hat{b}_{13} = \frac{1}{N_1}\sum_i Z_i\hat{e}_i(1-\hat{e}_i)^2(Y_i-\hat{\mu}_1)\tilde{X}_i, \qquad \hat{b}_{23} = \frac{1}{N_0}\sum_i(1-Z_i)\hat{e}_i^2(1-\hat{e}_i)(Y_i-\hat{\mu}_0)\tilde{X}_i.$$

Hence, we summarize the estimators for the asymptotic variance of $\hat{\tau}^{\text{OW}}_{\text{RD}}, \hat{\tau}^{\text{OW}}_{\text{RR}}, \hat{\tau}^{\text{OW}}_{\text{OR}}$ in the following equations,

$$\text{Var}(\hat{\tau}^{\text{OW}}) = \frac{1}{N}\left[\hat{V}^{\text{UNADJ}} - \hat{v}_1^T\left\{\frac{1}{N}\sum_{i=1}^N\hat{e}_i(1-\hat{e}_i)\tilde{X}_i^T\tilde{X}_i\right\}^{-1}(2\hat{v}_1 - \hat{v}_2)\right], \tag{1}$$

where

$$\hat{V}^{\text{UNADJ}} = \left\{\frac{1}{N}\sum_{i=1}^N\hat{e}_i(1-\hat{e}_i)\right\}^{-1}\left(\frac{\hat{E}_1^2}{N_1}\sum_{i=1}^N Z_i\hat{e}_i(1-\hat{e}_i)^2(Y_i-\hat{\mu}_1)^2 + \frac{\hat{E}_0^2}{N_0}\sum_{i=1}^N(1-Z_i)\hat{e}_i^2(1-\hat{e}_i)(Y_i-\hat{\mu}_0)^2\right),$$
$$\hat{v}_1 = \left\{\frac{1}{N}\sum_{i=1}^N\hat{e}_i(1-\hat{e}_i)\right\}^{-1}\left(\frac{\hat{E}_1}{N_1}\sum_{i=1}^N Z_i\hat{e}_i^2(1-\hat{e}_i)(Y_i-\hat{\mu}_1)^2\tilde{X}_i + \frac{\hat{E}_0}{N_0}\sum_{i=1}^N(1-Z_i)\hat{e}_i(1-\hat{e}_i)^2(Y_i-\hat{\mu}_0)^2\tilde{X}_i\right),$$
$$\hat{v}_2 = \left\{\frac{1}{N}\sum_{i=1}^N\hat{e}_i(1-\hat{e}_i)\right\}^{-1}\left(\frac{\hat{E}_1}{N_1}\sum_{i=1}^N Z_i\hat{e}_i(1-\hat{e}_i)^2(Y_i-\hat{\mu}_1)^2\tilde{X}_i + \frac{\hat{E}_0}{N_0}\sum_{i=1}^N(1-Z_i)\hat{e}_i^2(1-\hat{e}_i)(Y_i-\hat{\mu}_0)^2\tilde{X}_i\right),$$

and $\hat{E}_k$ depends on the estimands. For $\hat{\tau}^{\text{OW}}_{\text{RD}}$, we have $\hat{E}_k = 1$; for $\hat{\tau}^{\text{OW}}_{\text{RR}}$, we set $\hat{E}_k = \hat{\mu}_k^{-1}$; for $\hat{\tau}^{\text{OW}}_{\text{OR}}$, we use $\hat{E}_k = \hat{\mu}_k^{-1}(1-\hat{\mu}_k)^{-1}$ with $k = 0, 1$.

APPENDIX C: VARIANCE ESTIMATOR FOR $\hat{\tau}^{\text{AIPW}}$

In this appendix, we provide the details on how to derive the variance estimator for $\hat{\tau}^{\text{AIPW}}$ in the main text. Let $\mu_1(X_i;\alpha_1)$, $\mu_0(X_i;\alpha_0)$ be the outcome surface for treated and control samples respectively, with α_1, α_0 being the regression parameters. Suppose $\hat{\alpha}_1$, $\hat{\alpha}_0$ are the MLEs that solve the score functions $\sum_{i=1}^N Z_i S_1(Y_i, X_i;\alpha_1) = 0$ and $\sum_{i=1}^N (1-Z_i) S_0(Y_i, X_i;\alpha_0) = 0$. We resume our notation and let $e(X_i;\theta)$ be the propensity score, $\hat{\theta}$ be the parameters and $S_\theta(X_i;\theta)$ be the corresponding score function. Recall that $\hat{\tau}^{\text{AIPW}}$ takes the following form:

$$\hat{\tau}^{\text{AIPW}} = \hat{\mu}_1^{\text{AIPW}} - \hat{\mu}_0^{\text{AIPW}} = \frac{1}{N}\sum_{i=1}^N \left\{ \frac{Z_i Y_i}{\hat{e}_i} - \frac{(Z_i - \hat{e}_i)\hat{\mu}_1(X_i)}{\hat{e}_i} \right\} - \left\{ \frac{(1-Z_i)Y_i}{1-\hat{e}_i} + \frac{(Z_i - \hat{e}_i)\hat{\mu}_0(X_i)}{1-\hat{e}_i} \right\},$$

Let $\lambda = (\nu_1, \nu_0, \alpha_0, \alpha_1, \theta)$ and $\hat{\lambda} = (\hat{\nu}_1, \hat{\nu}_0, \hat{\alpha}_0, \hat{\alpha}_1, \hat{\theta})$. Note that $\hat{\lambda}$ is the solution for λ in the equations below:

$$\sum_{i=1}^N \Psi_i = \sum_{i=1}^N \begin{bmatrix} \nu_1 - \{Z_i Y_i - (Z_i - e_i)\mu_1(X_i;\alpha_1)\}/e_i \\ \nu_0 - \{(1-Z_i)Y_i + (Z_i - e_i)\mu_0(X_i;\alpha_0)\}/(1-e_i) \\ Z_i S_1(Y_i, X_i;\alpha_1) \\ (1-Z_i) S_0(Y_i, X_i;\alpha_0) \\ S_\theta(X_i;\theta) \end{bmatrix} = 0.$$

The asymptotic covariance of $\hat{\lambda}$ can be obtained via M-estimation theory, which equals $A^{-1}BA^T$, with $A = -E(\partial\Psi_i/\partial\lambda)$, $B = E(\Psi_i \Psi_i^T)$. In practice, we use plug-in method to estimate A, B. We can express $\hat{\tau}^{\text{AIPW}}$ with the solution $\hat{\lambda}$ as $\hat{\tau}^{\text{AIPW}} = \hat{\nu}_1 - \hat{\nu}_0$. Next, we can calculate the asymptotic variance of $\hat{\tau}^{\text{AIPW}}$ based on the asymptotic covariance of $\hat{\lambda}$ and the delta method. Similarly, it is straightforward to obtain the estimator for risk ratio estimator $\hat{\tau}^{\text{AIPW}}_{\text{RR}} = \log\left(\hat{\nu}_1/\hat{\nu}_0\right)$ and odds ratio estimator $\hat{\tau}^{\text{AIPW}}_{\text{OR}} = \log\left(\hat{\nu}_1/(1-\hat{\nu}_1)\right) - \log\left(\hat{\nu}_0/(1-\hat{\nu}_0)\right)$, as in Web Appendix B.

APPENDIX D: ADDITIONAL SIMULATIONS WITH BINARY OUTCOMES

D.1. Simulation Design

We conduct a second set of simulations where the outcomes are generated from a generalized linear model. Specifically, we assume the potential outcome follows a logistic regression model (model 3): for $z = 0, 1$,

$$\text{logit}\{\Pr(Y_i(z) = 1)\} = \eta + z\alpha + X_i^T \beta_0 + zX_i^T \beta_1, \quad i = 1, 2, \ldots, N, \tag{2}$$

where X_i denotes the vector of $p = 10$ baseline covariates simulated as in Section 4.1 in the main manuscript, and the parameter η represents the prevalence of the outcomes in the control arm, i.e., $u \approx \Pr\{Y_i(0) = 1\} = 1/(1 + \exp(-\eta))$. We specify the main effects $\beta_0 = b_0 \times (1, 1, 2, 2, 4, 4, 8, 8, 16, 16)^T$, where b_0 is chosen to be the same value used in Section 4.1 for continuous outcomes. For the covariate-by-treatment interactions, we set $\beta_1 = b_1 \times (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$ and examine scenarios with $b_1 = 0$ and $b_1 = 0.75$, with the latter representing strong treatment effect heterogeneity. Similarly, we set the true treatment effect to be zero $\tau = 0$. For the randomization probability r, we examine both balanced assignment with $r = 0.5$ and unbalanced assignment with $r = 0.7$. We vary the sample size N from 50 to 500 to represent both small and large sample sizes. We vary the value of η such that the baseline prevalence $u \in \{0.5, 0.3, 0.2, 0.1\}$, representing common to rare outcomes. It is expected that the regression adjustment becomes less stable with rare outcomes, while propensity score weighting estimators are less affected[2].

Under each scenario, we simulate 2000 data replicates, and compare five estimators, $\hat{\tau}^{\text{UNADJ}}$, $\hat{\tau}^{\text{IPW}}$, $\hat{\tau}^{\text{LR}}$, $\hat{\tau}^{\text{AIPW}}$, $\hat{\tau}^{\text{OW}}$, for binary outcomes. The unadjusted estimator is the nonparametric difference-in-mean estimator. For the IPW and OW estimators, we fit a propensity score model by regressing the treatment on the main effects of the baseline covariates X_i. With a slight abuse of acronym, in this Section we will use the abbreviation 'LR' to represent logistic regression. For this estimator, we fit the logistic outcome model with main effects of treatment and covariates, along with their interactions, as in $\text{logit}\{\Pr(Y_i = 1)\} = \delta + Z_i\kappa + X_i^T\xi_0 + Z_iX_i^T\xi_1$. The group means μ_0, μ_1 are estimated by standardization (i.e. the basic form of the g-formula[3]),

$$\hat{\mu}_0^{\text{LR}} = \frac{1}{N}\sum_{i=1}^N \frac{\exp(\hat{\delta} + X_i^T\hat{\xi}_0)}{1 + \exp(\hat{\delta} + X_i^T\hat{\xi}_0)}, \qquad \hat{\mu}_1^{\text{LR}} = \frac{1}{N}\sum_{i=1}^N \frac{\exp(\hat{\delta} + \hat{\kappa} + X_i^T\hat{\xi}_0 + X_i^T\hat{\xi}_1)}{1 + \exp(\hat{\delta} + \hat{\kappa} + X_i^T\hat{\xi}_0 + X_i^T\hat{\xi}_1)}. \tag{3}$$

The estimated group means are then used to calculate risk difference τ_{RD}, log risk ratio τ_{RR} and log odds ratio τ_{OR}. For the AIPW estimator, we estimate $\hat{\mu}_1^{\text{AIPW}}$ and $\hat{\mu}_0^{\text{AIPW}}$ as defined in equation (18) of the main text, except that $\hat{\mu}_z(X_i) = \hat{E}[Y_i|X_i, Z_i = z]$ is now the prediction from the above logistic outcome model. The ratio estimands are then estimated following equation (10) of the main text.

Because the bias of all these approaches is close to zero, we focus on the relative efficiency of the adjusted estimator compared to the unadjusted in estimating the three estimands. We also examine the performance of the variance and normality-based confidence interval estimators. For the LR estimator, we use the Huber-White variance, and then derive the large-sample variance of $\hat{\tau}_{\text{RD}}^{\text{LR}}$, $\hat{\tau}_{\text{RR}}^{\text{LR}}$ and $\hat{\tau}_{\text{OR}}^{\text{LR}}$ using the delta method. For IPW, we use the sandwich variance of Williamson et al.[2]; for OW, we use the sandwich variance proposed in Section 3.3 of the main text. Details of the variance calculation for the AIPW estimator is given in Web Appendix C.

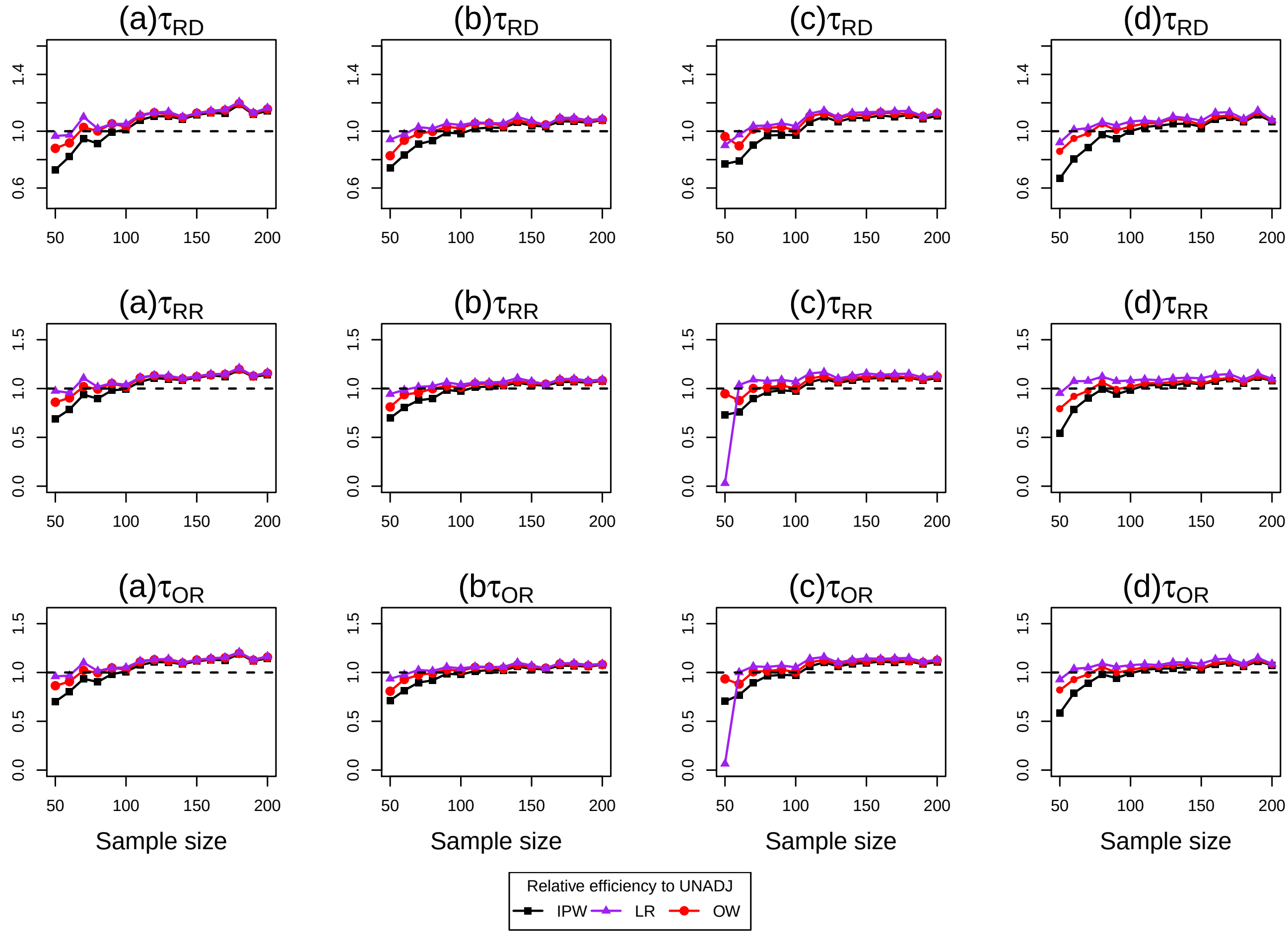
To explore the performance of estimators under model misspecification, we also repeat the simulations by considering a data generating process with additional covariate interaction terms (model 4): for $z = 0, 1$,

$$\text{logit}\{\Pr(Y_i(z) = 1)\} = \eta + z\alpha + X_i^T \beta_0 + zX_i\beta_1 + X_{i,\text{int}}^T\gamma, \quad i = 1, 2, \ldots, N, \tag{4}$$

which can be viewed as the binary analogy of model 2 defined in equation (19) of the main text. When the data are generated using model 4, we will examine the performance of a misspecified logistic regression ignoring the interaction terms $X_{i,\text{int}}$. Similarly, for IPW, OW and AIPW, the propensity score model will also ignore the interaction terms $X_{i,\text{int}}$.

D.2. Results on Efficiency of Point Estimators

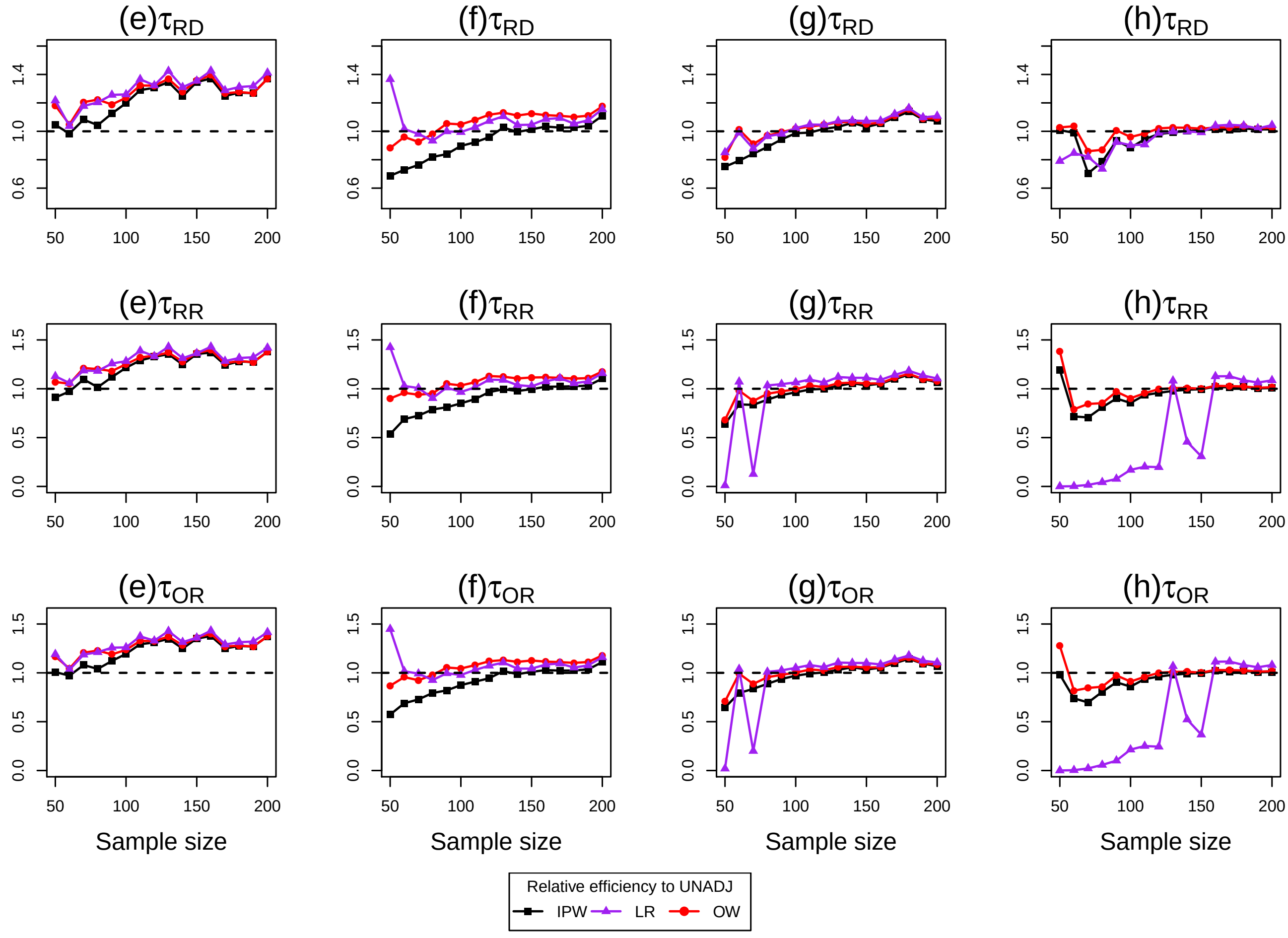
Within the range of sample sizes we considered, the potential efficiency gain using the covariate-adjusted estimators over the unadjusted estimator is *at most modest* for binary outcomes. Web Figure 1 presents the relative efficiency results. Because the finite-sample performance of AIPW is generally driven by the outcome regression component, we mainly focus on interpreting the comparisons between IPW, LR and OW. In column (a), where the outcome is common and the data are generated from model 3, $\hat{\tau}^{\text{IPW}}$, $\hat{\tau}^{\text{LR}}$ or $\hat{\tau}^{\text{OW}}$ become more efficient than $\hat{\tau}^{\text{UNADJ}}$ only when N is greater than 80. Because the true outcome model is used in model fitting, LR is slightly more efficient than OW and IPW but the difference quickly diminishes as N increases.



WEB FIGURE 1 The relative efficiency of $\hat{\tau}^{\text{IPW}}$,$\hat{\tau}^{\text{LR}}$,$\hat{\tau}^{\text{AIPW}}$ and $\hat{\tau}^{\text{OW}}$ relative to $\hat{\tau}^{\text{UNADJ}}$ for estimating τ_{RD}, τ_{RR}, τ_{OR}, when (a) $u = 0.5$ and the outcome model is correctly specified (b) $u = 0.5$ and the outcome model is misspecified (c) $u = 0.3$, and the outcome model is correctly specified (d) $u = 0.3$ and the outcome model is misspecified. A larger value of relative efficiency corresponds to a more efficient estimator.

The comparison results are similar when the outcome is generated from model 4 (column (b) and (d)). In addition, when the prevalence of the outcome decreases to around 30% (column (c)), the covariate-adjusted estimators become more efficient than the unadjusted estimator when $N > 100$. In this case, the correctly-specified LR estimator may become unstable in estimating the two ratio estimands when N is as small as 50, while both OW and IPW are not subject to such concerns because they do not attempt to estimate an outcome model.

Web Figure 2 presents the relative efficiency results in four additional scenarios. In the presence of strong treatment effect heterogeneity (column (e)), the covariate-adjusted estimators, LR and OW, improve over the unadjusted estimator even with a small sample size $N = 50$. In this case, the efficiency of LR and OW is almost identical across the range of sample size we examined. In contrast to the continuous outcome simulations, the LR estimator may become more efficient than OW and IPW with unbalanced randomization ($r = 0.7$) and $N \leq 80$ (column (f)). However, when the outcome becomes rare (column (g) and (h)), the OW and IPW estimators are more stable than LR. In these scenarios, the LR estimates can be quite variable, leading to dramatic efficiency loss even compared with the unadjusted estimator. With further investigation, we find that the LR estimator frequently run into numerical issues and fails to converge under rare outcomes. This non-convergence issue under rare outcomes also adversely affects the efficiency of the AIPW estimator. Web Table 4 summarizes the number of times that the logistic regression fails to converge as a function of sample size and prevalence of the outcome under the control condition. For instance, when the outcome is rare ($u = 0.1$), the logistic regression fails to converge more than half of the times even when



WEB FIGURE 2 The relative efficiency of $\hat{\tau}^{\text{IPW}}$, $\hat{\tau}^{\text{LR}}$, $\hat{\tau}^{\text{AIPW}}$ and $\hat{\tau}^{\text{OW}}$ relative to $\hat{\tau}^{\text{UNADJ}}$ for estimating $\tau_{\text{RD}}, \tau_{\text{RR}}, \tau_{\text{OR}}$, when (e) $u = 0.5$, $b_1 = 0.75$, $r = 0.5$ and the outcome model is correctly specified (f) $u = 0.5$, $b_1 = 0$, $r = 0.7$ and the outcome model is misspecified (g) $u = 0.2$,$b_1 = 0$, $r = 0.5$, and the outcome model is correctly specified (h) $u = 0.1$, $b_1 = 0$,$r = 0.5$, and the outcome model is correctly specified.

$N = 100$. Finally, for binary outcomes, the difference in efficiency between the adjusted estimators is more pronounced when N does not exceed 200, and becomes trivial when $N = 500$.

To summarize, we conclude that for binary outcomes

(i) covariate adjustment improves efficiency most likely when the sample size is at least 100, except in the presence of large treatment effect heterogeneity where there is efficiency gain even with $N = 50$.

(ii) the OW estimator is uniformly more efficient in finite samples than IPW and should be the preferred propensity score weighting estimator in randomized trials.

(iii) although correctly-specified outcome regression is slightly more efficient than OW in the ideal case with a non-rare outcome, in small samples regression adjustment is generally unstable when the prevalence of outcome decreases.

(iv) the efficiency of AIPW is mainly driven by the outcome regression component, and the instability of the outcome model may also lead to an inefficient AIPW estimator in finite-samples.

D.3. Results on Variance and Interval Estimators

For $N \in \{50, 100, 200, 500\}$, Web Table 2 and 3 further summarize the accuracy of the variance estimators and the empirical coverage rate of the corresponding interval estimator for each approach, in the scenarios presented in Web Figure 1 and 2. The

Williamson's variance estimator for IPW and the sandwich variance for AIPW frequently underestimate the true variance for all three estimands, so that the associated confidence interval shows under-coverage, especially when the sample size does not exceed 100. From a hypothesis testing point of view, as we are setting the average causal effect to be null, the results suggest the risk of type I error inflation using IPW or AIPW. Both LR and OW generally improve upon IPW and AIPW by maintaining closer to nominal coverage rate, with a few exceptions. For example, we notice that the Huber-White variance for logistic regression can be unstable and biased towards zero, leading to under-coverage. On the other hand, the proposed sandwich variance for OW is always close to the true variance regardless of the target estimand. Likewise, the OW interval estimator demonstrates improved performance over IPW, LR and AIPW, and maintains close to nominal coverage even in small samples with rare outcomes, where outcome regression frequently fails to converge.

To summarize, we conclude that for binary outcomes

(i) the Williamson's variance estimator for IPW and the sandwich variance for AIPW frequently underestimate the true variance for all three estimands.

(ii) the Huber-White variance for logistic regression can be unstable, and may have large bias in small samples with rare outcomes.

(iii) the proposed sandwich variance for OW is always close to the true variance regardless of the target estimand, and the OW interval estimator demonstrates close to nominal coverage even in small samples with rare outcomes.

APPENDIX E: PROGRAMMING CODE

In this appendix, we include the details to reproduce all results within the paper. Please download the codebase from https://github.com/zengshx777/OWRCT_codes_package with

R Scripts

`Main_RCT_Continuous.R`	Run simulations with continuous outcome.
`Main_RCT_Binary.R`	Run simulations with binary outcome.
`Crude.R`	Function implements $\hat{\tau}^{\text{UNADJ}}$.
`IPWC.R`	Function implements $\hat{\tau}^{\text{IPW}}$.
`LinearR.R`	Function implements $\hat{\tau}^{\text{LR}}$.
`PS_AIPW.R`	Function implements $\hat{\tau}^{\text{AIPW}}$.
`OW.R`	Function implements $\hat{\tau}^{\text{OW}}$.
`plot_cont.R`	Visualize continuous simulation results, produce Figure 1 in main text.
`plot_bin.R`	Visualize binary simulation results, produce Web Figure 1,2.
`table_produce.R`	Summarize all results, produce Table 1 in the main text, Web Table 1,2,3,4.
`example.R`	Simple demo for running simulations.
`all_jobs.sh`	Bash script to run all simulations.

To replicate the simulation results in the paper, the simplest way is to run `all_jobs.sh` after setting the code package as the working directory. The results will be automatically saved in folders ‘cont’ and ‘bin’.

APPENDIX F: WEB TABLES

Web Table 1 summarizes the full simulation results with continuous outcomes. we consider the following scenarios:

1. $r = 0.5, b_1 = 0$, model is correctly specified, corresponding to scenario (a) in Figure 1 of the main text.
2. $r = 0.5, b_1 = 0.25$, model is correctly specified.
3. $r = 0.5, b_1 = 0.5$, model is correctly specified.
4. $r = 0.5, b_1 = 0.75$, model is correctly specified, corresponding to scenario (b) in Figure 1 of the main text.
5. $r = 0.6, b_1 = 0$, model is correctly specified.
6. $r = 0.7, b_1 = 0$, model is correctly specified, corresponding to scenario (c) in Figure 1 of the main text.
7. $r = 0.5, b_1 = 0$, model is misspecified.
8. $r = 0.7, b_1 = 0$, model is misspecified, corresponding to scenario (d) in Figure 1 of the main text.

We include the additional numerical results for the simulations with binary outcomes in Web Table 2 and 3. For binary outcome, we consider the following scenarios,

1. $u = 0.5, r = 0.5, b_1 = 0$, model is correctly specified, corresponding to scenario (a) in Web Figure 2.
2. $u = 0.5, r = 0.5, b_1 = 0$, model is misspecified, corresponding to scenario (b) in Web Figure 2.
3. $u = 0.3, r = 0.5, b_1 = 0$, model is correctly specified, corresponding to scenario (c) in Web Figure 2.
4. $u = 0.3, r = 0.5, b_1 = 0$, model is misspecified, corresponding to scenario (d) in Web Figure 2.
5. $u = 0.5, r = 0.5, b_1 = 0.75$, model is correctly specified, corresponding to scenario (e) in Web Figure 3.
6. $u = 0.5, r = 0.7, b_1 = 0$, model is correctly specified, corresponding to scenario (f) in Web Figure 3.
7. $u = 0.2, r = 0.5, b_1 = 0$, model is correctly specified, corresponding to scenario (g) in Web Figure 3.
8. $u = 0.1, r = 0.5, b_1 = 0$, model is correctly specified, corresponding to scenario (h) in Web Figure 3.

For binary outcome, we also report in Web Table 4 the number of non-convergences for fitting logistic regression under different baseline outcome prevalence $u = 0.5, 0.3, 0.2, 0.1$.

□

WEB TABLE 1 The relative efficiency of each estimator compared to the unadjusted estimator, the ratio between the average estimated variance over Monte Carlo variance ({Est Var}/{MC Var}), and 95% coverage rate of IPW, LR, AIPW and OW estimators. The results are based on 2000 simulations with a continuous outcome. In the “correct specification" scenario, data are generated from model 1; in the "misspecification" scenario, data are generated from model 2. For each estimator, the same specification is used throughout, regardless of the data generating model.

Sample size	Relative efficiency				{Est Var}/{MC Var}				95% Coverage			
N	IPW	LR	AIPW	OW	IPW	LR	AIPW	OW	IPW	LR	AIPW	OW
					$r = 0.5$, $b_1 = 0$, correct specification							
50	1.621	2.126	2.042	2.451	1.001	0.866	0.668	1.343	0.936	0.933	0.885	0.967
100	2.238	2.475	2.399	2.548	0.898	0.961	0.799	1.116	0.938	0.944	0.914	0.955
200	2.927	2.987	2.984	3.007	0.951	0.996	0.927	1.051	0.946	0.949	0.938	0.956
500	2.985	3.004	2.995	3.006	0.963	0.987	0.959	1.000	0.944	0.949	0.942	0.952
					$r = 0.5$, $b_1 = 0.25$, correct specification							
50	1.910	2.792	2.606	2.905	1.141	0.711	0.684	1.562	0.946	0.899	0.887	0.972
100	2.968	3.575	3.481	3.489	0.988	0.811	0.896	1.295	0.954	0.925	0.928	0.968
200	3.640	3.864	3.855	3.794	0.932	0.754	0.923	1.079	0.940	0.912	0.933	0.956
500	3.801	3.814	3.814	3.791	0.947	0.735	0.940	0.992	0.945	0.907	0.945	0.950
					$r = 0.5$, $b_1 = 0.5$, correct specification							
50	1.635	2.894	2.781	2.755	1.021	0.463	0.769	1.530	0.936	0.822	0.910	0.970
100	3.084	3.917	3.835	3.546	0.984	0.510	0.977	1.291	0.942	0.840	0.944	0.968
200	3.187	3.410	3.406	3.287	0.924	0.446	0.936	1.061	0.944	0.802	0.942	0.956
500	3.730	3.809	3.810	3.717	1.037	0.477	1.049	1.085	0.957	0.818	0.960	0.962
					$r = 0.5$, $b_1 = 0.75$, correct specification							
50	1.715	3.043	2.972	2.570	0.991	0.286	0.816	1.383	0.935	0.712	0.918	0.967
100	2.679	3.279	3.253	3.003	0.931	0.280	0.917	1.168	0.942	0.710	0.934	0.966
200	2.979	3.220	3.215	3.023	0.967	0.278	0.995	1.075	0.951	0.697	0.949	0.964
500	3.337	3.425	3.426	3.338	0.995	0.273	1.013	1.037	0.943	0.696	0.945	0.954
					$r = 0.6$, $b_1 = 0$, correct specification							
50	1.415	1.686	1.605	2.418	1.041	0.745	0.617	1.377	0.938	0.913	0.883	0.959
100	2.042	2.378	2.290	2.521	0.889	0.942	0.784	1.104	0.944	0.941	0.915	0.956
200	2.777	2.926	2.896	2.981	0.987	1.027	0.947	1.078	0.949	0.950	0.940	0.953
500	2.898	2.939	2.939	2.950	0.976	0.994	0.969	1.003	0.953	0.953	0.949	0.953
					$r = 0.7$, $b_1 = 0$, correct specification							
50	1.056	0.036	0.036	2.270	1.060	0.014	0.026	1.184	0.938	0.779	0.816	0.931
100	1.825	2.439	2.311	2.935	0.914	0.858	0.717	1.039	0.946	0.921	0.897	0.923
200	2.474	2.706	2.679	2.874	0.971	0.931	0.857	0.963	0.948	0.944	0.927	0.935
500	2.641	2.743	2.738	2.809	0.922	0.912	0.887	0.925	0.940	0.936	0.934	0.938
					$r = 0.5$, $b_1 = 0$, misspecification							
50	1.009	1.093	0.986	1.299	0.773	0.768	0.598	0.900	0.908	0.915	0.870	0.933
100	1.371	1.502	1.379	1.549	0.805	0.954	0.779	0.924	0.924	0.946	0.921	0.942
200	1.526	1.567	1.516	1.592	0.897	0.965	0.888	0.925	0.938	0.953	0.936	0.944
500	1.576	1.587	1.569	1.595	0.913	0.937	0.911	0.912	0.943	0.949	0.944	0.941
					$r = 0.7$, $b_1 = 0$, misspecification							
50	0.896	0.009	0.009	1.468	0.843	0.005	0.009	0.857	0.904	0.777	0.808	0.906
100	1.096	1.258	1.152	1.533	0.724	0.754	0.637	0.837	0.911	0.903	0.878	0.917
200	1.390	1.457	1.398	1.570	0.861	0.894	0.816	0.898	0.929	0.938	0.920	0.933
500	1.591	1.632	1.612	1.648	0.980	1.003	0.976	0.981	0.948	0.949	0.948	0.949

WEB TABLE 2 The relative efficiency of each estimator compared to the unadjusted, the ratio between the average estimated variance ({Est Var}) over Monte Carlo variance ({MC Var}) and 95% coverage rate of IPW, LR, AIPW and OW estimators for binary outcomes. The scenarios correspond to Web Figure 1.

		Relative efficiency				{Est Var}/{MC Var}				95% Coverage			
	N	IPW	LR	AIPW	OW	IPW	LR	AIPW	OW	IPW	LR	AIPW	OW
		$u = 0.5$, $b_1 = 0$, $r = 0.5$, correct specification **(a)**											
	50	0.729	0.966	0.854	0.880	0.936	1.387	0.903	1.124	0.903	0.940	0.906	0.943
τ_{RD}	100	1.034	1.100	1.061	1.083	0.796	0.924	0.763	0.972	0.914	0.934	0.905	0.945
	200	1.152	1.159	1.149	1.158	0.985	1.049	0.967	1.164	0.944	0.953	0.945	0.961
	500	1.186	1.191	1.191	1.184	0.969	0.995	0.969	1.151	0.946	0.948	0.947	0.962
	50	0.690	0.976	0.832	0.860	0.910	1.372	0.870	1.097	0.924	0.966	0.926	0.964
τ_{RR}	100	1.038	1.104	1.062	1.090	0.803	0.927	0.766	0.979	0.922	0.942	0.915	0.953
	200	1.154	1.160	1.150	1.160	0.987	1.050	0.969	1.165	0.948	0.957	0.947	0.964
	500	1.189	1.193	1.194	1.186	0.971	0.996	0.970	1.152	0.950	0.952	0.949	0.965
	50	0.702	0.960	0.836	0.864	0.950	1.395	0.905	1.128	0.913	0.966	0.915	0.955
τ_{OR}	100	1.031	1.101	1.060	1.082	0.795	0.925	0.763	0.973	0.920	0.938	0.910	0.950
	200	1.153	1.160	1.150	1.159	0.985	1.050	0.968	1.164	0.946	0.954	0.946	0.963
	500	1.187	1.191	1.192	1.184	0.969	0.994	0.968	1.150	0.948	0.951	0.948	0.964
		$u = 0.5$, $b_1 = 0$, $r = 0.5$, misspecification **(b)**											
	50	0.742	0.942	0.848	0.827	0.888	1.225	0.825	0.996	0.887	0.943	0.902	0.921
τ_{RD}	100	0.971	1.057	1.002	1.033	0.813	0.996	0.799	0.976	0.913	0.945	0.911	0.937
	200	1.074	1.086	1.076	1.082	0.921	0.993	0.912	1.039	0.936	0.943	0.936	0.950
	500	1.100	1.106	1.105	1.100	0.962	0.993	0.963	1.088	0.948	0.950	0.948	0.957
	50	0.697	0.944	0.824	0.811	0.869	1.244	0.834	1.000	0.909	0.943	0.914	0.948
τ_{RR}	100	0.968	1.072	1.013	1.036	0.806	0.992	0.797	0.966	0.925	0.956	0.924	0.947
	200	1.071	1.084	1.075	1.078	0.913	0.983	0.903	1.029	0.940	0.948	0.940	0.955
	500	1.103	1.110	1.109	1.103	0.966	0.997	0.967	1.092	0.949	0.952	0.948	0.958
	50	0.714	0.936	0.831	0.808	0.890	1.231	0.826	0.997	0.902	0.950	0.909	0.943
τ_{OR}	100	0.966	1.058	1.001	1.031	0.810	0.995	0.797	0.973	0.919	0.951	0.920	0.944
	200	1.075	1.087	1.077	1.083	0.921	0.992	0.911	1.039	0.938	0.947	0.938	0.953
	500	1.100	1.107	1.106	1.101	0.962	0.993	0.963	1.088	0.949	0.951	0.948	0.958
		$u = 0.3$, $b_1 = 0$, $r = 0.5$, correct specification **(c)**											
	50	0.797	0.946	0.899	0.942	0.915	1.369	0.892	1.141	0.896	0.944	0.892	0.937
τ_{RD}	100	1.002	1.044	1.021	1.043	0.852	1.138	0.814	1.015	0.925	0.951	0.914	0.945
	200	1.123	1.124	1.116	1.130	0.976	1.154	0.952	1.131	0.942	0.960	0.940	0.957
	500	1.187	1.201	1.198	1.188	1.014	1.147	1.014	1.185	0.951	0.964	0.951	0.966
	50	0.758	0.034	0.004	0.938	1.004	0.051	0.004	1.241	0.919	0.964	0.917	0.971
τ_{RR}	100	1.010	1.070	1.041	1.043	0.859	1.173	0.818	1.019	0.936	0.965	0.929	0.956
	200	1.124	1.132	1.122	1.129	0.962	1.148	0.939	1.114	0.949	0.968	0.945	0.962
	500	1.189	1.204	1.201	1.189	1.007	1.141	1.007	1.176	0.954	0.966	0.955	0.968
	50	0.748	0.073	0.008	0.924	1.013	0.112	0.009	1.225	0.915	0.959	0.917	0.958
τ_{OR}	100	1.005	1.057	1.031	1.043	0.855	1.158	0.816	1.019	0.931	0.961	0.922	0.952
	200	1.124	1.129	1.120	1.130	0.968	1.152	0.945	1.123	0.946	0.965	0.942	0.960
	500	1.188	1.203	1.200	1.189	1.011	1.144	1.010	1.181	0.952	0.964	0.953	0.967
		$u = 0.3$, $b_1 = 0$, $r = 0.5$, misspecification **(d)**											
	50	0.667	0.921	0.687	0.858	0.924	1.471	0.889	1.204	0.883	0.976	0.943	0.926
τ_{RD}	100	0.950	1.021	0.977	0.989	0.859	1.196	0.837	1.019	0.918	0.958	0.912	0.948
	200	1.126	1.139	1.133	1.126	0.946	1.156	0.931	1.072	0.940	0.963	0.938	0.953
	500	1.116	1.137	1.132	1.118	1.031	1.209	1.029	1.183	0.951	0.966	0.952	0.962
	50	0.543	0.952	0.630	0.795	0.885	1.515	1.039	1.189	0.905	0.986	0.953	0.959
τ_{RR}	100	0.941	1.041	0.993	0.975	0.843	1.202	0.822	1.000	0.932	0.971	0.923	0.961
	200	1.127	1.147	1.142	1.123	0.949	1.170	0.934	1.074	0.946	0.969	0.939	0.958
	500	1.115	1.139	1.135	1.117	1.028	1.208	1.026	1.178	0.953	0.968	0.954	0.964
	50	0.583	0.928	0.634	0.818	0.917	1.498	0.999	1.196	0.900	0.981	0.953	0.945
τ_{OR}	100	0.944	1.031	0.985	0.981	0.851	1.201	0.829	1.010	0.926	0.965	0.920	0.953
	200	1.127	1.143	1.138	1.125	0.947	1.163	0.932	1.074	0.940	0.966	0.940	0.957
	500	1.116	1.138	1.134	1.118	1.029	1.209	1.027	1.181	0.952	0.967	0.954	0.963

WEB TABLE 3 The relative efficiency of each estimator compared to the unadjusted, the ratio between the average estimated variance ({Est Var}) over Monte Carlo variance ({MC Var}) and 95% coverage rate of IPW, LR, AIPW and OW estimators for binary outcomes. The scenarios correspond to Web Figure 2.

		Relative efficiency				{Est Var}/{MC Var}				95% Coverage			
	N	IPW	LR	AIPW	OW	IPW	LR	AIPW	OW	IPW	LR	AIPW	OW
	$u = 0.5$, $b_1 = 0.75$, $r = 0.5$, correct specification **(e)**												
	50	1.046	1.217	1.129	1.181	0.905	1.151	0.707	1.066	0.895	0.857	0.829	0.944
τ_{RD}	100	1.248	1.294	1.281	1.305	0.945	1.028	0.855	1.298	0.931	0.939	0.921	0.968
	200	1.365	1.420	1.411	1.367	0.988	1.014	0.966	1.353	0.945	0.947	0.941	0.976
	500	1.329	1.381	1.380	1.328	0.899	0.871	0.897	1.246	0.940	0.934	0.938	0.973
	50	0.910	1.128	0.989	1.069	0.866	1.066	0.634	0.998	0.916	0.914	0.857	0.966
τ_{RR}	100	1.257	1.283	1.272	1.305	0.959	1.022	0.855	1.306	0.938	0.940	0.933	0.976
	200	1.358	1.416	1.408	1.361	0.986	1.012	0.966	1.347	0.946	0.951	0.950	0.981
	500	1.330	1.384	1.383	1.329	0.899	0.871	0.898	1.244	0.940	0.936	0.940	0.974
	50	1.009	1.191	1.107	1.168	0.912	1.136	0.704	1.089	0.909	0.857	0.857	0.957
τ_{OR}	100	1.246	1.291	1.276	1.305	0.944	1.027	0.851	1.295	0.938	0.946	0.924	0.973
	200	1.368	1.425	1.416	1.371	0.988	1.015	0.966	1.353	0.945	0.948	0.944	0.979
	500	1.330	1.383	1.381	1.329	0.900	0.871	0.898	1.246	0.942	0.935	0.940	0.974
	$u = 0.5$, $b_1 = 0$, $r = 0.7$, correct specification **(f)**												
	50	0.619	1.379	1.328	0.882	0.871	18.187	0.560	0.803	0.848	0.917	0.836	0.901
τ_{RD}	100	0.902	0.999	0.956	1.026	0.850	0.971	0.760	1.134	0.898	0.949	0.905	0.951
	200	1.017	1.047	1.033	1.081	0.849	0.898	0.808	1.122	0.920	0.935	0.913	0.960
	500	1.165	1.180	1.173	1.189	0.981	1.007	0.972	1.281	0.945	0.948	0.944	0.973
	50	0.447	1.547	1.472	0.791	0.806	10.114	0.546	0.702	0.877	0.911	0.859	0.935
τ_{RR}	100	0.872	0.987	0.938	1.025	0.843	0.963	0.757	1.136	0.916	0.954	0.922	0.961
	200	1.017	1.052	1.038	1.085	0.843	0.893	0.804	1.112	0.928	0.941	0.920	0.963
	500	1.166	1.180	1.174	1.190	0.977	1.002	0.968	1.274	0.952	0.952	0.949	0.974
	50	0.489	1.512	1.450	0.816	0.881	5.454	0.545	0.728	0.892	0.915	0.842	0.928
τ_{OR}	100	0.888	0.996	0.949	1.026	0.848	0.972	0.759	1.134	0.908	0.956	0.914	0.958
	200	1.015	1.046	1.032	1.081	0.848	0.897	0.807	1.120	0.929	0.941	0.919	0.962
	500	1.166	1.181	1.174	1.189	0.981	1.007	0.972	1.280	0.946	0.951	0.946	0.973
	$u = 0.2$, $b_1 = 0$, $r = 0.5$, correct specification **(g)**												
	50	0.755	0.806	0.758	0.807	0.738	1.093	0.689	0.863	0.887	0.915	0.851	0.917
τ_{RD}	100	0.904	0.968	0.952	0.938	0.869	1.485	0.863	1.008	0.916	0.965	0.920	0.933
	200	1.103	1.129	1.120	1.114	0.925	1.296	0.918	1.048	0.938	0.973	0.933	0.955
	500	1.103	1.108	1.108	1.102	0.988	1.256	0.979	1.123	0.949	0.971	0.948	0.960
	50	0.642	0.010	0.001	0.671	0.868	0.017	0.002	1.034	0.914	0.957	0.900	0.973
τ_{RR}	100	0.908	1.028	1.004	0.933	0.860	1.532	0.856	0.997	0.925	0.977	0.939	0.952
	200	1.102	1.147	1.137	1.110	0.899	1.283	0.895	1.017	0.946	0.978	0.944	0.962
	500	1.097	1.104	1.104	1.096	0.983	1.253	0.973	1.116	0.949	0.977	0.949	0.964
	50	0.649	0.020	0.003	0.698	0.861	0.033	0.003	1.030	0.906	0.957	0.900	0.960
τ_{OR}	100	0.906	1.009	0.987	0.934	0.863	1.522	0.858	1.002	0.923	0.974	0.930	0.949
	200	1.103	1.142	1.133	1.112	0.907	1.289	0.903	1.028	0.943	0.976	0.938	0.960
	500	1.099	1.105	1.106	1.098	0.985	1.255	0.975	1.118	0.949	0.976	0.948	0.962
	$u = 0.1$, $b_1 = 0$, $r = 0.5$, correct specification **(h)**												
	50	0.995	0.800	0.785	1.032	0.238	0.255	0.193	0.277	0.888	0.440	0.417	0.912
τ_{RD}	100	0.892	0.881	0.852	0.939	1.064	2.224	0.996	1.194	0.922	0.980	0.947	0.940
	200	1.038	1.056	1.044	1.054	0.958	1.878	0.948	1.042	0.938	0.991	0.942	0.947
	500	1.076	1.101	1.100	1.078	0.985	1.577	0.989	1.068	0.949	0.988	0.947	0.954
	50	0.570	0.001	0.000	1.057	0.608	0.001	0.000	1.201	0.939	0.375	1.000	0.991
τ_{RR}	100	0.868	0.979	0.940	0.893	1.089	2.348	1.024	1.232	0.944	0.994	0.952	0.972
	200	1.052	1.132	1.115	1.065	0.938	1.910	0.940	1.019	0.949	0.994	0.948	0.957
	500	1.073	1.101	1.098	1.074	0.976	1.565	0.975	1.058	0.951	0.990	0.951	0.960
	50	0.610	0.002	0.000	1.078	0.685	0.002	0.000	1.335	0.928	0.375	1.000	0.985
τ_{OR}	100	0.872	0.960	0.923	0.901	1.085	2.329	1.018	1.226	0.938	0.993	0.948	0.965
	200	1.050	1.121	1.105	1.063	0.941	1.909	0.941	1.024	0.948	0.993	0.945	0.954
	500	1.074	1.101	1.098	1.075	0.977	1.568	0.977	1.060	0.951	0.990	0.950	0.958

WEB TABLE 4 Number of times that the logistic regression fails to converge given different outcome prevalence $u \in \{0.5, 0.3, 0.2, 0.1\}$ and sample sizes $N \in [50, 200]$.

N	$u = 0.5$	$u = 0.3$	$u = 0.2$	$u = 0.1$
50	1649	1802	1905	1975
60	1025	1320	1699	1947
70	525	823	1245	1829
80	207	433	834	1659
90	84	194	527	1393
100	34	89	307	1199
110	5	41	159	941
120	5	20	88	684
130	0	3	44	498
140	0	0	17	331
150	0	1	10	251
160	0	0	11	176
170	0	0	2	117
180	0	0	0	85
190	0	0	0	45
200	0	0	0	38