Supporting Information for "A Spatiotemporal Quantile Regression Model for Emergency Department Expenditures"

Brian Neelon*, **Fan Li, Lane F. Burgette and Sara E. Benjamin Neelon**

*neelon@musc.edu

Web Appendix: MCMC Algorithm

(1) **Update $\boldsymbol{\beta}_\tau$**: Given a $\mathrm{N}_p(\boldsymbol{\mu}_{0,\tau}, \boldsymbol{\Sigma}_{0,\tau})$ prior, the full conditional for the vector of fixed effect and spline coefficients is normal with mean $\boldsymbol{\mu}_{\boldsymbol{\beta}_\tau}$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\beta}_\tau}$ given by

$$\boldsymbol{\Sigma}_{\boldsymbol{\beta}_\tau} = \Big[\frac{\delta^2}{\zeta_\tau}\boldsymbol{X}^T\boldsymbol{D}^{-1}\boldsymbol{X} + \boldsymbol{\Sigma}_{0,\tau}^{-1}\Big]^{-1} \text{ and}$$

$$\boldsymbol{\mu}_{\boldsymbol{\beta}_\tau} = \boldsymbol{\Sigma}_{\boldsymbol{\beta}_\tau}\Big[\frac{\delta^2}{\zeta_\tau}\boldsymbol{X}^T\boldsymbol{D}^{-1}(\boldsymbol{y} - \xi_\tau\boldsymbol{w} - \boldsymbol{Z}_\phi\boldsymbol{\phi}_\tau - \boldsymbol{Z}_\psi\boldsymbol{\psi}_\tau - \boldsymbol{Z}_{\tilde{\boldsymbol{\theta}}}\tilde{\boldsymbol{\theta}}_\tau) + \boldsymbol{\Sigma}_{0,\tau}^{-1}\boldsymbol{\mu}_{0,\tau}\Big].$$

Here, $\boldsymbol{X}$ is an $N \times p$ design matrix for the fixed effects and spline bases; $\boldsymbol{Z}_{\boldsymbol{\phi}}$ is an $N \times n$ design matrix for the spatial effects $\boldsymbol{\phi}_\tau$; $\boldsymbol{Z}_{\boldsymbol{\psi}}$ is an $N \times J$ design matrix for the temporal effects $\boldsymbol{\psi}_\tau$; $\boldsymbol{Z}_{\tilde{\boldsymbol{\theta}}}$ is an $N \times (nJ)$ design matrix for the vectorized space-time interactions $\tilde{\boldsymbol{\theta}}_\tau$; $\boldsymbol{D}$ is a $N \times N$ diagonal matrix of latent weights w_{ijk}; $\boldsymbol{w}$ is the vector of latent weights; and ξ_τ and ζ_τ are defined in equation (10).

(2) **Update $\boldsymbol{\phi}_\tau$**: We jointly update the vector of spatial effects from $\mathrm{N}_n(\boldsymbol{\mu}_{\boldsymbol{\phi}_\tau}, \boldsymbol{\Sigma}_{\boldsymbol{\phi}_\tau})$ full conditional, where

$$\boldsymbol{\Sigma}_{\boldsymbol{\phi}_\tau} = \Big[\frac{\delta^2}{\zeta_\tau}\boldsymbol{Z}_\phi^T\boldsymbol{D}^{-1}\boldsymbol{Z}_\phi + \frac{1}{\sigma^2_{\phi,\tau}}\boldsymbol{Q}\Big]^{-1} \text{ and}$$

$$\boldsymbol{\mu}_{\boldsymbol{\phi}_\tau} = \boldsymbol{\Sigma}_{\boldsymbol{\phi}_\tau}\Big[\frac{\delta^2}{\zeta_\tau}\boldsymbol{Z}_\phi^T\boldsymbol{D}^{-1}(\boldsymbol{y} - \xi_\tau\boldsymbol{w} - \boldsymbol{X}\boldsymbol{\beta}_\tau - \boldsymbol{Z}_\psi\boldsymbol{\psi}_\tau - \boldsymbol{Z}_{\tilde{\boldsymbol{\theta}}}\tilde{\boldsymbol{\theta}}_\tau)\Big].$$

Here, $\sigma^2_{\phi,\tau}$ is the spatial conditional variance, and $\boldsymbol{Q} = \boldsymbol{M} - \boldsymbol{A}$ is defined in equation (6). At the end of the step, $\boldsymbol{\phi}_\tau$ should be centered to satisfy $\sum_{i=1}^n \phi_{i,\tau} = 0$.

(3) **Update $\boldsymbol{\psi}_\tau$**: Update the $J \times 1$ vector $\boldsymbol{\psi}_\tau$ from its $\mathrm{N}_J(\boldsymbol{\mu}_{\boldsymbol{\psi}_\tau}, \boldsymbol{\Sigma}_{\boldsymbol{\psi}_\tau})$ full conditional, where

$$\boldsymbol{\Sigma}_{\boldsymbol{\psi}_\tau} = \Big[\frac{\delta^2}{\zeta_\tau}\boldsymbol{Z}_\psi^T\boldsymbol{D}^{-1}\boldsymbol{Z}_\psi + \frac{1}{\sigma^2_{\psi,\tau}}\boldsymbol{K}\Big]^{-1} \text{ and}$$

$$\boldsymbol{\mu}_{\boldsymbol{\psi}_\tau} = \boldsymbol{\Sigma}_{\boldsymbol{\psi}_\tau}\Big[\frac{\delta^2}{\zeta_\tau}\boldsymbol{Z}_\psi^T\boldsymbol{D}^{-1}(\boldsymbol{y} - \xi_\tau\boldsymbol{w} - \boldsymbol{X}\boldsymbol{\beta}_\tau - \boldsymbol{Z}_\phi\boldsymbol{\phi}_\tau - \boldsymbol{Z}_{\tilde{\boldsymbol{\theta}}}\tilde{\boldsymbol{\theta}}_\tau)\Big].$$

Here, $\sigma^2_{\psi,\tau}$ is the temporal variance, and $\boldsymbol{K}$ is defined in expression (8). Center $\boldsymbol{\psi}_\tau$ to ensure $\sum_{j=1}^J \psi_{j,\tau} = 0$.

(4) **Update** $\tilde{\boldsymbol{\theta}}_\tau$: Assuming the prior given in equation (9), we update the long vector $\tilde{\boldsymbol{\theta}}_\tau$ from its $\mathrm{N}_{nJ}(\boldsymbol{\mu}_{\tilde{\boldsymbol{\theta}}_\tau}, \boldsymbol{\Sigma}_{\tilde{\boldsymbol{\theta}}_\tau})$ full conditional given by

$$\boldsymbol{\Sigma}_{\tilde{\boldsymbol{\theta}}_\tau} = \Big[\frac{\delta^2}{\zeta_\tau}\boldsymbol{Z}_{\tilde{\boldsymbol{\theta}}}^T\boldsymbol{D}^{-1}\boldsymbol{Z}_{\tilde{\boldsymbol{\theta}}} + \frac{1}{\sigma^2_{\theta,\tau}}(\boldsymbol{K}\otimes\boldsymbol{Q})\Big]^{-1} \text{ and}$$

$$\boldsymbol{\mu}_{\tilde{\boldsymbol{\theta}}_\tau} = \boldsymbol{\Sigma}_{\tilde{\boldsymbol{\theta}}_\tau}\Big[\frac{\delta^2}{\zeta_\tau}\boldsymbol{Z}_{\tilde{\boldsymbol{\theta}}}^T\boldsymbol{D}^{-1}(\boldsymbol{y} - \xi_\tau\boldsymbol{w} - \boldsymbol{X}\boldsymbol{\beta}_\tau - \boldsymbol{Z}_\phi\boldsymbol{\phi}_\tau - \boldsymbol{Z}_\psi\boldsymbol{\psi}_\tau)\Big],$$

and subsequently store the posterior samples as an $n \times J$ matrix, $\boldsymbol{\theta}_\tau$. In cases where one (or both) of $\boldsymbol{Q}$ or $\boldsymbol{K}$ retains a small dimension, the J columns of $\boldsymbol{\theta}_\tau$ can be updated separately to improve computation efficiency. At the end of the step, center the updated matrix $\boldsymbol{\theta}_\tau$ both by row and column to ensure $\sum_{i=1}^n \theta_{ij} = 0\ \forall j$ and $\sum_{j=1}^J \theta_{ij} = 0\ \forall i$.

(5) **Update** $\sigma^2_{\phi,\tau}$: Assuming an $\mathrm{IG}(c,d)$ prior, draw $\sigma^2_{\phi,\tau}$ from its $\mathrm{IG}(c^*, d^*)$ full conditional, where $c^* = c + (n-1)/2$, $d^* = d + \boldsymbol{\phi}_\tau^T\boldsymbol{Q}\boldsymbol{\phi}_\tau/2$, n is the number of areal units and $\boldsymbol{Q}$ is defined in equation (6).

(6) **Update** $\sigma^2_{\psi,\tau}$: Assuming an $\mathrm{IG}(e,f)$ prior, draw $\sigma^2_{\psi,\tau}$ from its $\mathrm{IG}(e^*, f^*)$ full conditional, where $e^* = e + (J-1)/2$, $f^* = f + \boldsymbol{\psi}^T\boldsymbol{K}\boldsymbol{\psi}/2$, J is the number of years, and $\boldsymbol{K}$ is defined in equation (8).

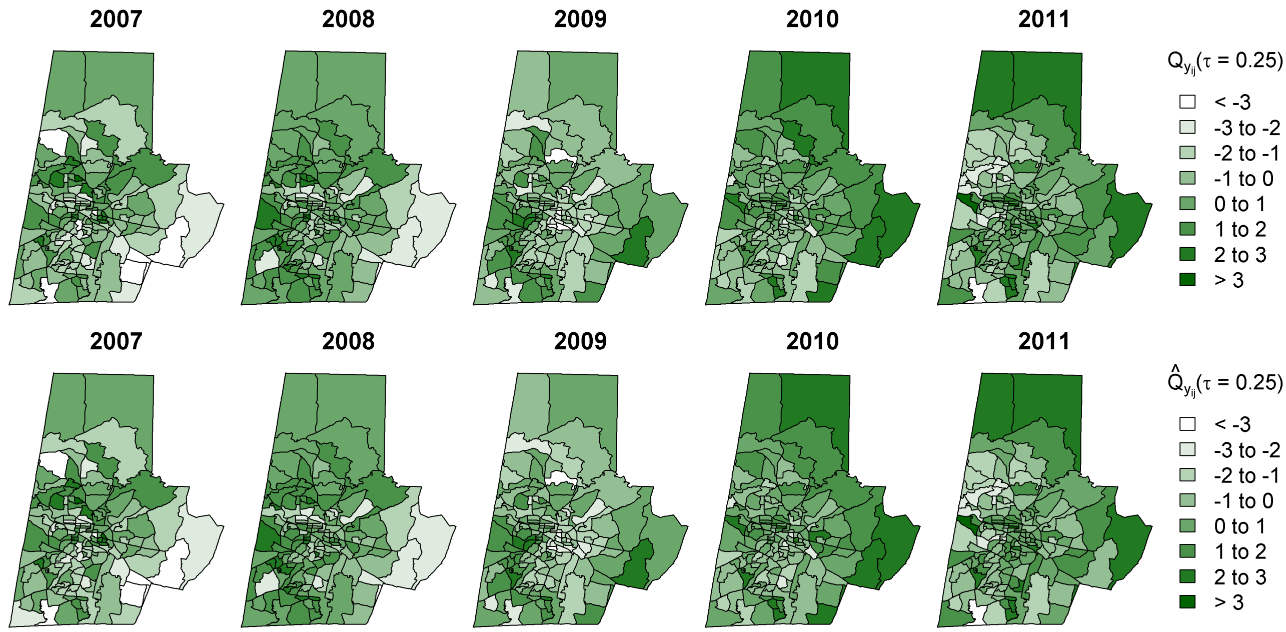
(7) **Update** $\sigma^2_{\theta,\tau}$: Assuming an $\mathrm{IG}(g,h)$ prior, draw $\sigma^2_{\theta,\tau}$ from its $\mathrm{IG}(g^*, h^*)$ full conditional, where $g^* = g + (n-1)(J-1)/2$, $h^* = h + \tilde{\boldsymbol{\theta}}_\tau^T(\boldsymbol{K}\otimes\boldsymbol{Q})\tilde{\boldsymbol{\theta}}_\tau/2$.

(8) **Update** w_{ijk}: For all i, j, k, sample the inverse latent weight w_{ijk}^{-1} independently from its inverse-Gaussian full conditional

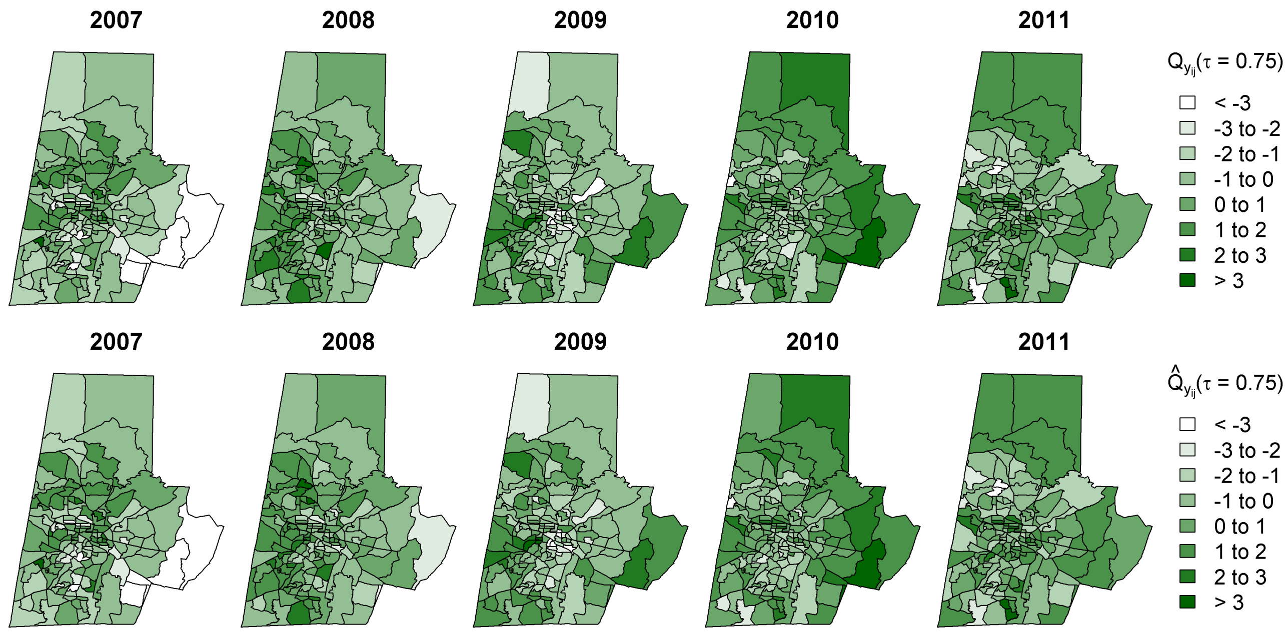
$$w_{ijk}^{-1} \sim \mathrm{InvGauss}\Big(\frac{\sqrt{\xi_\tau^2 + 2\zeta_\tau}}{|y_{ijk} - \boldsymbol{x}_{ijk}^T\boldsymbol{\beta}_\tau - \phi_{i,\tau} - \psi_{j,\tau} - \theta_{ij,\tau}|}, \frac{\delta^2(\xi_\tau^2 + 2\zeta_\tau)}{\zeta_\tau}\Big).$$

(9) **Update** δ^2: Assuming a $\mathrm{Ga}(a,b)$ prior, draw the ALD precision parameter from its $\mathrm{Ga}(a^*, b^*)$ full conditional, where $a^* = a + \frac{3}{2}N$, $b^* = b + \sum_{i=1}^n\sum_{j=1}^J\sum_{k=1}^{n_{ij}}(2\zeta_\tau w_{ijk})^{-1}(y_{ijk} - \xi_\tau w_{ijk} - \boldsymbol{x}_{ijk}^T\boldsymbol{\beta}_\tau - \phi_{i,\tau} - \psi_{j,\tau} - \theta_{ij,\tau})^2 + \sum_{i=1}^n\sum_{j=1}^J\sum_{k=1}^{n_{ij}} w_{ijk}$; $N = \sum_{i=1}^n\sum_{j=1}^J n_{ij}$ is the total number of patients. Alternatively, the partially collapsed Gibbs sampler of Reed and Yu (2009) integrates out the latent weights to yield a slightly different gamma full conditional, where $a^* = a + N$, $b^* = b + \sum_{i=1}^n\sum_{j=1}^J\sum_{k=1}^{n_{ij}}[y_{ijk} - \boldsymbol{x}_{ijk}^T\boldsymbol{\beta}_\tau - \phi_{i,\tau} - \psi_{j,\tau} - \theta_{ij,\tau}][\tau - \mathbb{I}(y_{ijk} - \boldsymbol{x}_{ijk}^T\boldsymbol{\beta}_\tau - \phi_{i,\tau} - \psi_{j,\tau} - \theta_{ij,\tau} < 0)]$, and $\mathbb{I}(.)$ denotes the indicator function.

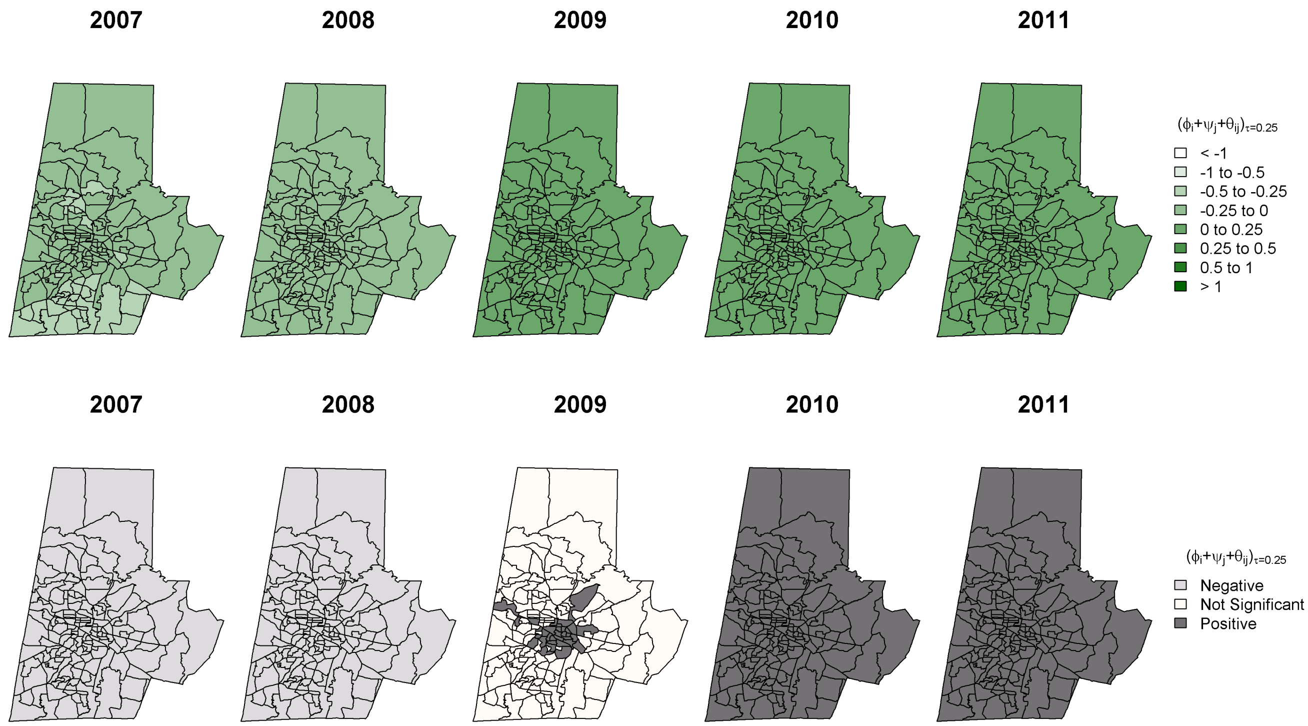
Web Figures



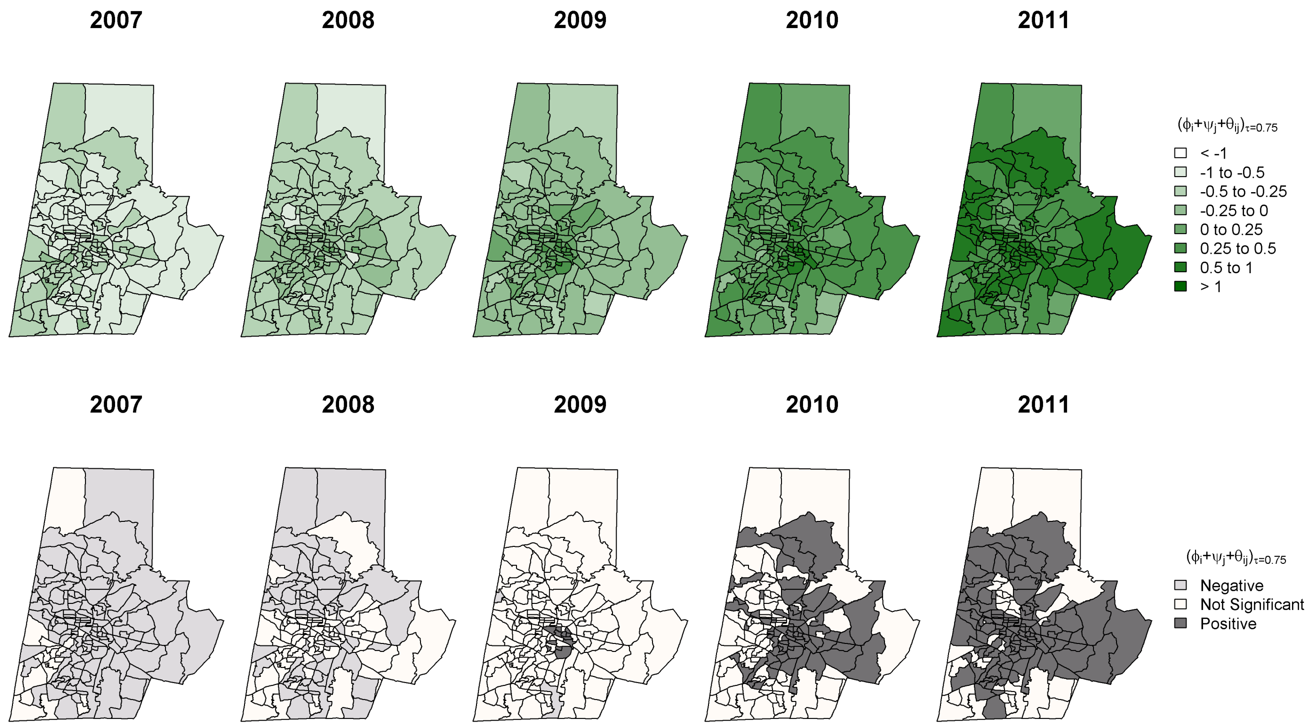
Web Figure 1: Observed and estimated 25th-quantile maps for the illustrative example. Upper panel: observed 25th quantiles (mean centered). Lower panel: estimated 25th quantiles (mean centered).



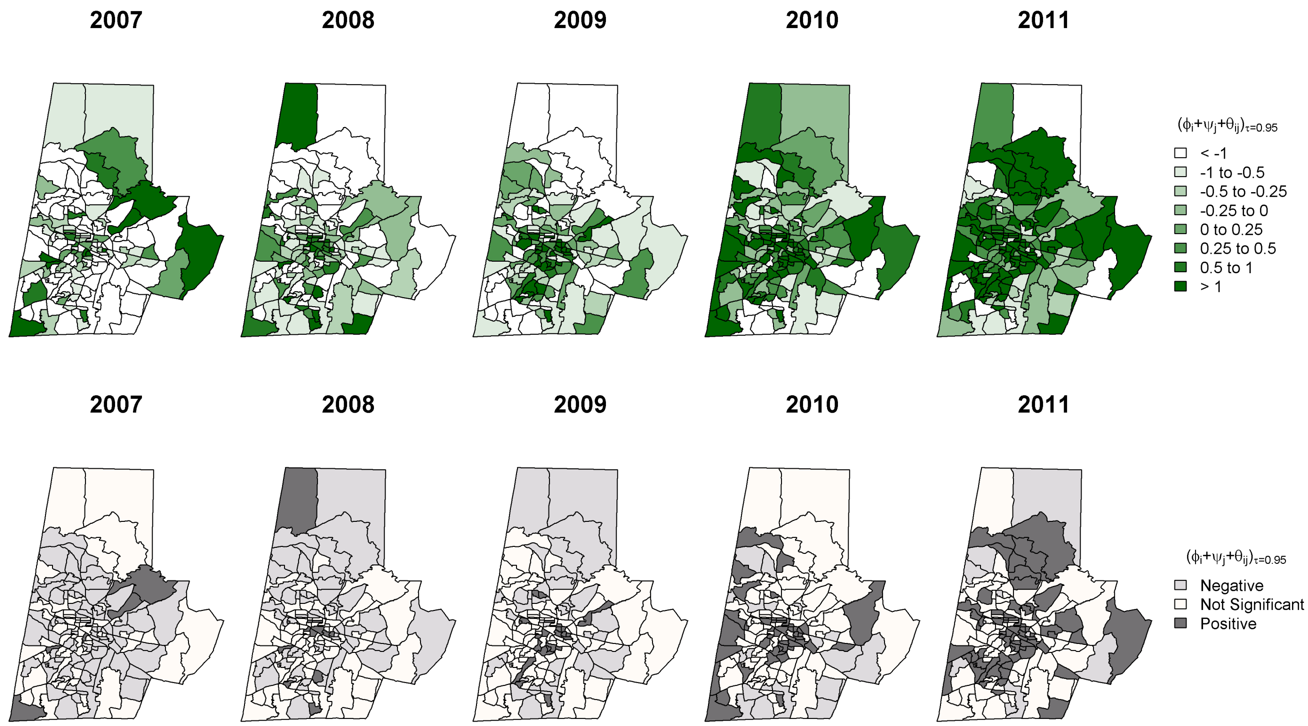
Web Figure 2: Observed and estimated 75th-quantile maps for the illustrative example. Upper panel: observed 75th quantiles (mean centered). Lower panel: estimated 75th quantiles (mean centered).



Web Figure 3: Predicted spatiotemporal random effects for the 25th quantile model in the DSR analysis. Upper panel: predicted spatiotemporal effects ($1000s). Lower panel: block groups with significant negative effects in light gray, significant positive effects in dark gray, and nonsignificant effects in white.



Web Figure 4: Predicted spatiotemporal random effects for the 75th quantile model in the DSR analysis. Upper panel: predicted spatiotemporal effects ($1000s). Lower panel: block groups with significant negative effects in light gray, significant positive effects in dark gray, and nonsignificant effects in white.



Web Figure 5: Predicted spatiotemporal random effects for the 95th quantile model in the DSR analysis. Upper panel: predicted spatiotemporal effects ($1000s). Lower panel: block groups with significant negative effects in light gray, significant positive effects in dark gray, and nonsignificant effects in white.

References

Reed C. and Yu K. (2009). A partially collapsed Gibbs sampler for Bayesian quantile regression. Technical Report. University of Brunel, School of Information Systems, Computing and Mathematics. Available from: http://bura.brunel.ac.uk/handle/2438/3593.